# A practical stratified importance sampling strategy for extremely rare event estimation in high dimensional systems

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**Abstract**

Estimating extremely rare events in complex systems represents a significant and persistent challenge in reliability analysis. This difficulty is further compounded in high-dimensional settings, where the target probability mass becomes exceedingly difficult to capture, introducing new obstacles to accurate estimation. To address these challenges, this paper proposes a practical and adaptive stratified importance sampling (IS) strategy for reliability analysis involving high-dimensional inputs and extremely rare events. The proposed method makes several key contributions. First, it theoretically demonstrates that within existing stratified IS frameworks, the accuracy of estimating intermediate failure events does not impact the final rare-event probability estimate. Leveraging this insight, we design an adaptive stratified strategy that prioritizes rapid progression toward the failure domain over precise intermediate event estimation, thereby avoiding the computational inefficiency inherent in exhaustively exploring intermediate failure regions.

At the core of our approach is a novel sampling density construction. We develop a mixture IS density based on a bi-domain K-means clustering technique, integrated with variance-enlarged sampling. This design enables the method to not only converge efficiently toward the failure region but also accurately capture diverse and challenging failure modes, including discontinuous patterns and those with sharp gradient transitions. To mitigate the significant estimation errors that can arise from minor deviations in failure information in high-dimensional problems, an optimization mechanism is innovatively introduced in the final sampling layer. This allows for the accurate identification of the most probable failure information in high-dimensional space, leading to precise estimation of extremely rare events in high-dimensional systems. The effectiveness of the proposed method is validated through a series of benchmark problems in reliability analysis, encompassing high dimensionality, small failure probabilities, and complex failure domains. Results consistently show that our method outperforms existing approaches in both capturing varied failure modes and delivering efficient, high-precision estimates, particularly for high-dimensional systems.

**Keywords:** Rare event estimation; High dimensional system; Importance sampling.

## Introduction

Structural reliability analysis aims to evaluate the probability that a structure fails under uncertainties associated with random loads, material properties, geometric dimensions, and environmental factors. It is a fundamental problem in structural safety assessment and reliability-based design (Cavalieri et al., 2017). For complex engineering structures, failure events typically correspond to small-probability regions in a high dimensional random input space. In particular, under high-reliability design requirements, the target failure probability often reaches the order of $10^{-6}$ or even lower (Cheng et al., 2022; Li et al., 2023). In such cases, failure samples are extremely rare under the original probability distribution. The high dimensional nature of the problem further exacerbates the curse of dimensionality, resulting in highly sparse effective samples and making the geometric characteristics of the failure domain difficult to characterize (Cheng et al., 2023).

Therefore, accurately and stably estimating small failure probabilities in high dimensional problems with limited performance function evaluations remains a key challenge in structural reliability analysis.

For subsequent methodological development, the basic mathematical formulation of a structural reliability problem is first introduced. Assume that the performance function of a structure is defined as：

$$Y = g(\boldsymbol{X}) = g(X_1, X_2, \cdots, X_n) \tag{1}$$

where $\boldsymbol{X} = \{X_1, X_2, \cdots, X_n\}^T$ is an $n$-dimensional random input variable. The limit-state equation $g(\boldsymbol{x}) = 0$ divides the input variable space into the failure domain $F = \{\boldsymbol{x} : g(\boldsymbol{x}) \le 0\}$ and the safe domain $S = \{\boldsymbol{x} : g(\boldsymbol{x}) > 0\}$. The structural failure probability can be expressed as

$$P_f = \int_F f_{\boldsymbol{X}}(\boldsymbol{x}) d\boldsymbol{x} \tag{2}$$

where $f_{\boldsymbol{X}}(\boldsymbol{x})$ is the joint probability density function (PDF) of $\boldsymbol{X}$. When the input variables are mutually independent $f_{\boldsymbol{X}}(\boldsymbol{x}) = \prod_{i=1}^{n} f_{X_i}(x_i)$,where $f_{X_i}(x_i)(i = 1, 2, \cdots, n)$ is the marginal PDF of the input variable $X_i$.

By introducing the failure-domain indicator function $I_F(\boldsymbol{x})$, the failure probability in Eq. (2) can be rewritten as:

$$P_f = \int_F f_{\boldsymbol{X}}(\boldsymbol{x}) d\boldsymbol{x} = \int_{R^n} I_F(\boldsymbol{x}) f_{\boldsymbol{X}}(\boldsymbol{x}) d\boldsymbol{x} = E[I_F(\boldsymbol{x})] \tag{3}$$

where,

$$I_F(\boldsymbol{x}) = \begin{cases} 1 & \boldsymbol{x} \in F \\ 0 & \boldsymbol{x} \notin F \end{cases} \tag{4}$$

Existing methods for structural reliability analysis can generally be classified into three categories: approximate analytical methods, numerical simulation methods, and surrogate-model-based methods. Approximate analytical methods mainly include the First-Order Reliability Method (FORM) (Hasofer, 1974) and the Second-Order Reliability Method (SORM) (Der Kiureghian et al., 1987). These methods usually perform first-order or second-order Taylor expansions of the performance function at the Most Probable Point (MPP), thereby replacing the actual limit-state surface with a local approximation. Because a large number of direct evaluations of the original performance function are avoided, these methods usually offer relatively high computational efficiency. However, when the limit-state surface is strongly nonlinear or implicit, or when multiple failure domains exist, local-expansion-based approximations may not accurately characterize the actual failure boundary, thereby affecting the accuracy of reliability estimates.

In contrast, numerical simulation methods have become important tools in structural reliability analysis because of their clear principles and broad applicability. The fundamental method is Monte Carlo Simulation (MCS). MCS does not require additional assumptions regarding the performance function and is theoretically applicable to reliability problems with arbitrary forms. However, when small failure probabilities are considered, it usually requires a very large number of samples and therefore incurs a high computational cost. For this reason, it is often used as a reference benchmark for other efficient algorithms (Song and Kawai, 2023). To improve the efficiency of numerical simulation, various methods have been developed, including Importance Sampling (IS) (Tabandeh et al., 2022), Subset Simulation (SuS) (Au and Beck, 2001; Papaioannou et al., 2015), Line Sampling (Pradlwarter et al., 2007), and Directional Sampling (Zhang et al., 2020).

The core idea of IS is to construct an importance sampling probability density function (IS-PDF), thereby enabling the sampling process to generate more samples with relatively large contributions to the failure probability. Classical IS often improves the occurrence probability of failure samples by shifting the importance sampling center (ISC) toward the vicinity of the MPP. On this basis, various improved strategies, such as Meta-IS (Dubourg et al., 2013), have been developed. In contrast, SuS represents a small failure probability as the product of a series of conditional failure probabilities, thereby effectively avoiding the exponential increase in sample size as the failure probability decreases. SuS therefore exhibits relatively high efficiency in small failure probability analysis. According to the method used to generate conditional samples, SuS can be further classified into traditional SuS methods based on Markov chain Monte Carlo (MCMC) and SS-IS methods based on importance sampling (Song and Lu, 2008).

Although substantial progress has been achieved using existing methods, reliability analysis remains challenging when the focus is further narrowed to high dimensional problems involving extremely small failure probabilities. When the failure probability is extremely small, the failure domain is often far from the high-probability-mass region of the input random variables, resulting in an extremely limited number of failure samples. As the dimensionality further increases, the dispersion of samples in the high dimensional space significantly weakens the ability of conventional stochastic simulation methods to identify critical failure regions. To improve the efficiency and stability of estimating extremely small failure probabilities in high dimensional settings, researchers have proposed many specialized methods. Representative methods include generalized subset simulation (GSS), sequential importance sampling (SIS), sequential directional importance sampling (SDIS), subset adaptive importance sampling (SAIS), and AL-MRBF-iCE-m*.

GSS progressively relaxes the failure threshold and enlarges input variance to approach extremely rare failure regions; it further uses coordinate rotation in high-dimensional settings (Cheng et al., 2022). SIS constructs a sequence of intermediate distributions toward an IS-PDF and uses conditional-sampling MCMC to reduce dimension sensitivity (Papaioannou et al., 2014).

SDIS (Cheng et al., 2023) represents an extremely small failure probability as a sequence of auxiliary failure probabilities. At the first level, global exploration is conducted using MCS in Cartesian coordinates. At subsequent levels, directional importance sampling is performed in polar coordinates, and the sampling process gradually approaches the actual failure domain through sequential resampling and move mechanisms. SAIS (Helal and Elvira, 2026) further combines a multi-proposal mixture distribution, adaptive threshold updating, and a historical-sample recycling mechanism to improve coverage of complex failure domains, particularly multimodal failure domains. Meanwhile, for computationally expensive engineering black-box models, AL-MRBF-iCE-m* (Yang et al., 2025) combines active learning surrogate models with dimension-reduced cross-entropy importance sampling to estimate extremely small failure probabilities while reducing the number of true model evaluations.

Overall, existing methods for high dimensional and extremely small failure probability problems commonly employ sequential or stratified simulation, variance reduction, or a combination of both to rapidly and accurately approach the failure domain, and substantial progress has been made in improving sampling efficiency. However, the performance of these methods usually depends strongly on the quality of the constructed intermediate distributions, the effectiveness of proposal distribution updating, and the ability to identify critical failure regions. When the failure domain exhibits multimodal, nonlinear, or complex geometric characteristics, how to stably and comprehensively capture all important failure regions remains a central challenge in extremely small failure probability estimation.

Taking the classical sequential simulation method of MCMC-based SuS as an example, samples at each level are usually generated through Markov chains and may therefore exhibit strong autocorrelation, which affects sample quality and estimation efficiency. For SS-IS methods that combine sequential simulation and variance reduction, the existing designs usually emphasize the accurate estimation of the conditional failure probability at each level. Therefore, a relatively large number of samples is allocated to each intermediate level to ensure statistical accuracy. However, from the estimation form of SS-IS, once the IS-PDF at the final level has been determined, the failure probability can be further expressed as a single-level importance sampling estimate with respect to this final-level density. This observation indicates that the role of the intermediate levels should not be understood solely as providing high-accuracy estimates of conditional probabilities at successive levels. More importantly, the intermediate levels provide an informational basis for constructing an effective final-level IS-PDF through sample screening, regional advancement, and sampling-center updating. In other words, if the accuracy requirement for estimating intermediate-level conditional failure probabilities can be appropriately relaxed, and more computational resources can be allocated to improving the quality of the final-level IS-PDF, the overall computational efficiency may be substantially improved without affecting the final failure probability estimate. Moreover, existing SS-IS methods often select ISC primarily based on PDF information without further correcting their relative positions with respect to the true limit-state surface. In high dimensional problems, the final IS-PDF is generally more sensitive to positional deviations of the ISCs. Even a slight shift of the center may weaken the ability of the IS-PDF to cover important failure regions and lead to reduced estimation stability and efficiency.

Based on the above analysis, this study focuses on high dimensional and extremely small failure probability problems. Starting from the SS-IS framework, it re-examines the distinct roles of the intermediate and final levels in failure probability estimation and accordingly develops a more efficient adaptive stratified importance sampling strategy. The central idea is to reduce the dependence on highly accurate estimates of intermediate-level conditional failure probabilities and shift the methodological focus toward the efficient construction of the final-level IS-PDF and the accurate identification of critical failure regions. In this way, the overall efficiency and robustness of extremely small failure probability estimation in high dimensional problems can be improved.

The remainder of this paper is organized as follows. The first part reveals, through an equivalent transformation of the failure probability estimator in SS-IS, that the estimation of failure probability in SS-IS can be expressed as an integral involving only the IS-PDF at the final level. The second part presents the underlying idea and detailed implementation procedure of the proposed method. The third part validates the accuracy, stability, and computational efficiency of the proposed method through three numerical examples involving complex failure domains, high dimensional linear problems with extremely small failure probabilities, and high dimensional nonlinear problems with extremely small failure probabilities.

## 1. Analysis for SS-IS method

The SuS method introduces a sequence of intermediate failure events $F_k = \{\boldsymbol{x} : g(\boldsymbol{x}) \le b_k\}(k = 1, 2, \cdots, m)$, where $b_1 > b_2 > \cdots > b_m = 0$ denotes a sequence of critical values. In this case, $F_1 \supset F_2 \supset \cdots \supset F_m = F$ and $F_k = \bigcap_{i=1}^{k} F_i$, and the final target failure probability is transformed into the product of a sequence of conditional failure probabilities. According to the multiplication theorem in probability theory and the inclusion relationship among the events, the formula shown in Eq. (5) can be obtained.

$$
\begin{aligned}
P_f &= P\{F\} = P\left\{\bigcap_{k=1}^{m} F_k\right\} = P\left\{F_m \mid \bigcap_{k=1}^{m-1} F_k\right\} \cdot P\left\{\bigcap_{k=1}^{m-1} F_k\right\} \\
&= P\{F_m \mid F_{m-1}\} \cdot P\left\{F_{m-1} \mid \bigcap_{k=1}^{m-2} F_k\right\} \cdot P\left\{\bigcap_{k=1}^{m-2} F_k\right\} \\
&\cdots\cdots \\
&= P\{F_1\} \cdot \prod_{k=2}^{m} P\{F_k \mid F_{k-1}\}
\end{aligned} \tag{5}
$$

For simplicity, let $P_1 = P\{F_1\}$ and $P_k = P\{F_k \mid F_{k-1}\}(k = 2,3,\cdots,m)$. Then the final failure probability can be expressed as:

$$
P_f = \prod_{k=1}^{m} P_k \tag{6}
$$

The term $P_1$ in Eq. (6) can be estimated by Eq. (7) as follows:

$$
\hat{P}_1 = \hat{P}\{F_1\} = \frac{1}{N_1}\sum_{j=1}^{N_1} I_{F_1}(\boldsymbol{x}_j^1) \tag{7}
$$

where $\{\boldsymbol{x}_j^1\}_{j=1}^{N_1}$ denotes the $N_1$ samples generated from the joint PDF $f_{\boldsymbol{X}}(\boldsymbol{x})$ of the input variables $\boldsymbol{X}$.

The conditional failure probability $P_k$ in Eq. (6) can be estimated by Eq. (8) as follows:

$$
\hat{P}_k = \hat{P}\{F_k \mid F_{k-1}\} = \frac{1}{N_k}\sum_{j=1}^{N_k} I_{F_k}(\boldsymbol{x}_j^k) \quad (k = 2,3,\cdots,m) \tag{8}
$$

where $\{\boldsymbol{x}_j^k\}_{j=1}^{N_k}$ denotes the $N_k$ samples generated from the conditional PDF $q_{\boldsymbol{X}}(\boldsymbol{x} \mid F_{k-1})$ of the input variables $\boldsymbol{X}$ shown in Eq. (9).

$$
q_{\boldsymbol{X}}(\boldsymbol{x} \mid F_{k-1}) = \frac{I_{F_{k-1}}(\boldsymbol{x}) f_{\boldsymbol{X}}(\boldsymbol{x})}{P\{F_{k-1}\}} \quad (k = 2,3,\cdots,m) \tag{9}
$$

Conditional samples following the conditional PDF $q_{\boldsymbol{X}}(\boldsymbol{x} \mid F_{k-1})$ can be generated directly by Monte Carlo sampling. However, this approach is inefficient. Therefore, the SuS method mainly uses MCMC to obtain conditional samples following $q_{\boldsymbol{X}}(\boldsymbol{x} \mid F_{k-1})$.

Although MCMC can effectively generate samples following $q_{\boldsymbol{X}}(\boldsymbol{x} \mid F_{k-1})$, the Markov-chain mechanism usually introduces a certain degree of correlation among the generated samples. To reduce sample correlation, the SS-IS method has been proposed (Song and Lu, 2008). This method efficiently obtains conditional failure samples through IS. Its basic procedure starts from the joint PDF of the input variables, sequentially constructs the IS-PDF function on each subset, estimates the corresponding conditional failure probability for each subset, and finally obtains the required estimate of the failure probability.

In SS-IS, $P_1$ is also estimated by Eq.(7), whereas the conditional failure probabilities $P_k = P\{F_k \mid F_{k-1}\}(k = 2,3,\cdots,m)$ is calculated using the IS method shown in Eq. (10):

$$
P_k = \int_{R^n} I_{F_k}(\boldsymbol{x}) \cdot q_{\boldsymbol{X}}(\boldsymbol{x} \mid F_{k-1}) d\boldsymbol{x} = \int_{R^n} \frac{I_{F_k}(\boldsymbol{x}) \cdot q_{\boldsymbol{X}}(\boldsymbol{x} \mid F_{k-1})}{h_{\boldsymbol{X}}^k(\boldsymbol{x})} h_{\boldsymbol{X}}^k(\boldsymbol{x}) d\boldsymbol{x} = E\left[\frac{I_{F_k}(\boldsymbol{x}) \cdot q_{\boldsymbol{X}}(\boldsymbol{x} \mid F_{k-1})}{h_{\boldsymbol{X}}^k(\boldsymbol{x})}\right] \tag{10}
$$

where $h_{\boldsymbol{X}}^k(\boldsymbol{x})$ is the IS-PDF of the $k$-th intermediate failure event. This IS-PDF takes the sample with the largest original PDF value among the conditional failure samples at level $k$-1 as the ISC and constructs its covariance matrix using the variance of the original PDF.

After drawing $N_k$ samples $\{\boldsymbol{x}_j^k\}_{j=1}^{N_k}$ of the input variables from the IS-PDF $h_{\boldsymbol{X}}^k(\boldsymbol{x})$, the estimate $\hat{P}_k$ of the conditional failure probabilities $P_k$ can be obtained as shown in Eq. (11).

$$\hat{P}_k = \frac{1}{N_k}\sum_{j=1}^{N_k}\frac{I_{F_k}(\boldsymbol{x}_j^k)q_{\boldsymbol{X}}(\boldsymbol{x}_j^k \mid F_{k-1})}{h_{\boldsymbol{X}}^k(\boldsymbol{x}_j^k)} \quad (k = 2,3,\cdots,m) \tag{11}$$

According to Eq. (10), the conditional failure probability at the $m$-th level, namely the final level, can be expressed as Eq. (12):

$$P_m = \int_{R^n}\frac{I_{F_m}(\boldsymbol{x})\cdot q_{\boldsymbol{X}}(\boldsymbol{x} \mid F_{m-1})}{h_{\boldsymbol{X}}^m(\boldsymbol{x})}h_{\boldsymbol{X}}^m(\boldsymbol{x})d\boldsymbol{x} \tag{12}$$

If the conditional failure probability $P_m$ at the $m$-th level is separated from Eq. (6) and the failure probability of the $k$-th intermediate failure event ($k$ = 2,3,...,$m$) is considered, which can be expressed as:

$$P\{F_k\} = P\left\{\bigcap_{i=1}^{k}F_i\right\} = P\{F_k \mid F_{k-1}\}P\{F_{k-1} \mid F_{k-2}\}\cdots P\{F_2 \mid F_1\} = \prod_{i=1}^{k}P_i \tag{13}$$

Thus, the following expression can be obtained:

$$\begin{aligned} P_f &= \prod_{k=1}^{m}P_k = \prod_{k=1}^{m-1}P_k \times P_m \\ &= \prod_{k=1}^{m-1}P_k \times \int_{R^n}\frac{I_{F_m}(\boldsymbol{x})I_{F_{m-1}}(\boldsymbol{x})f_{\boldsymbol{X}}(\boldsymbol{x})}{P\{F_{m-1}\}h_{\boldsymbol{X}}^m(\boldsymbol{x})}h_{\boldsymbol{X}}^m(\boldsymbol{x})d\boldsymbol{x} \\ &= \frac{\prod_{k=1}^{m-1}P_k}{P\{F_{m-1}\}} \times \int_{R^n}\frac{I_{F_m}(\boldsymbol{x})I_{F_{m-1}}(\boldsymbol{x})f_{\boldsymbol{X}}(\boldsymbol{x})}{h_{\boldsymbol{X}}^m(\boldsymbol{x})}h_{\boldsymbol{X}}^m(\boldsymbol{x})d\boldsymbol{x} \\ &= \int_{R^n}\frac{I_{F_m}(\boldsymbol{x})f_{\boldsymbol{X}}(\boldsymbol{x})}{h_{\boldsymbol{X}}^m(\boldsymbol{x})}h_{\boldsymbol{X}}^m(\boldsymbol{x})d\boldsymbol{x} \end{aligned} \tag{14}$$

Eq. (14) shows that the failure probability $P_f$ can be equivalently expressed as a single-level IS integral with respect to the $m$-th-level (final-level) IS-PDF $h_{\boldsymbol{X}}^m(\boldsymbol{x})$. Therefore, once the final-level IS-PDF $h_{\boldsymbol{X}}^m(\boldsymbol{x})$ is determined, the total failure probability can be estimated by a weighted integral under this PDF, and the estimator can be written as:

$$P_f = \frac{1}{N_m}\sum_{j=1}^{N_m}\frac{I_{F_m}(\boldsymbol{x}_j)f_{\boldsymbol{X}}(\boldsymbol{x}_j)}{h_{\boldsymbol{X}}^m(\boldsymbol{x}_j)} \tag{15}$$

where $N_m$ denotes the number of samples used in the final-level importance sampling.

Therefore, in the SS-IS method, the accuracy of the final failure probability estimate actually depends only on the construction quality of the IS-PDF in the final-level and has no direct relationship with the construction of the intermediate-level IS-PDFs or the estimation accuracy of the conditional failure probabilities. This indicates that SS-IS does not need to accurately estimate the conditional failure probability at every intermediate level. Instead, it only needs to gradually approach the true limit-state surface through level-by-level advancement, sample screening, and sampling-center updating, and then construct a high-quality IS-PDF at the final level.

It should also be noted that although the intermediate levels do not directly affect the accuracy of the final failure probability estimate, they still indirectly influence the accuracy, variance, and stability of the final estimate by affecting the construction quality of the final-level IS-PDF. Therefore, compared with traditional SS-IS, which emphasizes accurate level-by-level estimation of the intermediate conditional failure probabilities, this study focuses more on the role of intermediate levels in the process of approaching the limit-state surface, identifying potential important regions, and constructing a high-quality final IS-PDF. Based on the above derivation, this paper redesigns the construction strategy of the intermediate-level IS-PDF in SS-IS, so that intermediate samples can rapidly approach the limit-state surface, identify important failure regions, and support high-quality construction of the final-level IS-PDF.

## 2 A practical stratified IS strategy for extremely rare event estimation in high dimensional systems

To address the limitations of traditional SS-IS, namely that a single ISC cannot retain information from multiple potential important failure regions and that the traditional ISC updating strategy has limited stratified advancement efficiency, this paper proposes a practical stratified IS strategy. The strategy first enlarges the sampling variance of the first-level IS-PDF to enhance the initial exploration of potential failure regions. It then introduces K-means clustering to identify multiple candidate ISCs and construct a mixture IS-PDF for the intermediate levels. At the same time, a safe/failure bi-domain ISC screening mechanism, together with a corresponding stratified stopping criterion, is proposed to balance failure-region exploration and the efficiency of approaching the limit-state surface. Finally, in constructing the final-level IS-PDF, an optimization procedure is employed to obtain local optimal solutions to the maximization of the original PDF within failure domain. These local optimal solutions are then used as the ISCs for constructing the final-level IS-PDF, thereby improving the overall estimation accuracy and computational efficiency of the proposed method.

For ease of implementation, the proposed method is calculated in the standard normal space. For mutually independent non-standard normal random variables, marginal distribution transformation can be used. For correlated random variables, the Nataf transformation or Rosenblatt transformation can be adopted. To simplify the notation, this section directly assumes that all input variables $\boldsymbol{X}$ follow mutually independent standard normal distributions, and the corresponding performance function in the standard normal space is denoted by $g(\boldsymbol{X})$. The specific implementation steps of the proposed method are as follows:

*Stage 1: Searching for potential important failure regions*

1. **Initialization of the first-level IS-PDF and sample generation:** Determine the conditional failure sample ratio $p_0$ and the standard-deviation enlargement factor $\alpha = 3$ (Cheng et al., 2023) for the first level to obtain the first-level IS-PDF $h_{\boldsymbol{X}}^1(\boldsymbol{x}) = \prod_{r=1}^{n} \frac{1}{\alpha\sqrt{2\pi}} \exp(-\frac{x_r^2}{2\alpha^2})$. Then generate $N_1$ samples $\{\boldsymbol{x}_j^1\}_{j=1}^{N_1}$ following $h_{\boldsymbol{X}}^1(\boldsymbol{x})$ and set $k = 1$.
2. **Determination of the first-level conditional failure threshold:** Calculate the performance-function values $\{g(\boldsymbol{x}_j^1)\}_{j=1}^{N_1}$ of the $N_1$ samples. Sort these $N_1$ response values in descending order and denote the sorted results as $g(\boldsymbol{x}_{[1]}^1) > g(\boldsymbol{x}_{[2]}^1) > \cdots > g(\boldsymbol{x}_{[N_1]}^1)$, where $[\bullet]$ denotes the index after the response values are sorted in descending order, rather than the original sample index. The $\lceil (1-p_0)N_1 \rceil$-th response value is taken as the critical value $b_1$ of the intermediate failure event $F_1 = \{\boldsymbol{x} : g(\boldsymbol{x}) \le b_1\}$, where $b_1 = g(\boldsymbol{x}_{[\lceil (1-p_0)N_1 \rceil]}^1)$ and $\lceil \bullet \rceil$ denotes the upward rounding operator.
3. **Level-index update:** $k = k + 1$.
4. **Identification of potential important regions based on K-means:** Use K-means clustering to divide

the sample points $\{\mathbf{x}_j^{k-1}\}_{j=[\lceil(1-p_0)N_{k-1}\rceil]}^{[N_{k-1}]}$ falling into the intermediate failure domain $F_{k-1}$ into $K^*$ clusters, $\mathcal{C}_s^k, s=1,2,...,K^*$, where $N_{k-1}$ denotes the number of samples at level $k$-1 and $K^*$ denotes the number of clusters. Within each cluster, the conditional failure samples are further divided into two groups according to the indicator function $I_F(\mathbf{x})$ of the final failure domain, i.e., safe samples $\Omega_{KS,s}=\{\mathbf{x}\in\mathcal{C}_s^k \mid g(\mathbf{x})>0\}$ and failure samples $\Omega_{KF,s}=\{\mathbf{x}\in\mathcal{C}_s^k \mid g(\mathbf{x})\le 0\}$.

5. **Safe/failure bi-domain ISC screening:** If $\Omega_{KS,s}\neq\varnothing$, select the sample $\mathbf{x}_{KS,s}=\arg\min_{\mathbf{x}\in\Omega_{KS,s}}\phi_n(\mathbf{x})$ with the smallest PDF value $\phi_n(\mathbf{x})$ from $\Omega_{KS,s}$. If $\Omega_{KF,s}\neq\varnothing$, select the sample $\mathbf{x}_{KF,s}=\arg\max_{\mathbf{x}\in\Omega_{KF,s}}\phi_n(\mathbf{x})$ with the largest PDF value $\phi_n(\mathbf{x})$ from $\Omega_{KF,s}$, where $\phi_n(\mathbf{x})$ denotes the PDF of the standard normal distribution. By screening the $K^*$ clusters of samples, the ISCs obtained from the safe samples can be represented as $\{\mathbf{x}_{KS,j}^k\}_{j=1}^{N_{KS}}$, where $N_{KS}\le K^*$ denotes the number of ISCs obtained from the $K^*$ clusters of safe samples. The ISCs obtained from the failure samples can be represented as $\{\mathbf{x}_{KS,j}^k\}_{j=1}^{N_{KF}}$, where $N_{KF}\le K^*$ denotes the number of ISCs obtained from the $K^*$ clusters of failure samples.

6. **Construction of the ISC set:** Merge $\{\mathbf{x}_{KS,j}^k\}_{j=1}^{N_{KS}}$ and $\{\mathbf{x}_{KS,j}^k\}_{j=1}^{N_{KF}}$ to obtain the ISC set $\{\mathbf{x}_{\mathrm{ISC}_k,j}^k\}_{j=1}^{N_{\mathrm{ISC}_k}}$ of level $k$, where $N_{\mathrm{ISC}_K}=N_{KS}+N_{KF}\le 2K^*$ is the number of ISCs.

7. **Judgment of the stratified stopping criterion:** Determine whether $N_{KF}$ is equal to $K^*$. If $N_{KF}\neq K^*$, some of the $K^*$ clusters contain no failure samples and therefore do not yield corresponding failure ISC. This suggests that some potential important regions identified by K-means may not yet have reached the true failure domain. In this case, the samples should be further advanced toward the limit-state surface, and the algorithm proceeds to Step 8 for the next-level search. If $N_{KF}=K^*$, each of the $K^*$ clusters contains at least one failure sample and therefore yields a corresponding failure ISC. This indicates that all potential important regions represented by the current K-means clusters have reached the true failure domain. Consequently, the main potential failure regions are regarded as having been sufficiently explored under current condition. Therefore, set $m$ = $k$, and $b_m=0$, and proceed to Step 12 to construct the final-level IS-PDF.

8. **Construction of the next-level IS-PDF and generation of importance samples:** For each ISC from $\{\mathbf{x}_{\mathrm{ISC},i}^k\}_{i=1}^{N_{\mathrm{ISC}_k}}$, construct a Gaussian distribution $h_{\mathbf{x}_{\mathrm{ISC},i}^k}^k(\mathbf{x})$ using $\mathbf{x}_{\mathrm{ISC},i}^k$ as the ISC and the identity covariance matrix as the covariance matrix. Then construct the $k$-th-level IS-PDF using an equally weighted Gaussian mixture strategy as follows,

$$h_{\mathbf{X}}^k(\mathbf{x})=\frac{1}{N_{\mathrm{ISC}_k}}\sum_{i=1}^{N_{\mathrm{ISC}_k}} h_{\mathbf{x}_{\mathrm{ISC},i}^k}^k(\mathbf{x}) \tag{16}$$

This is because that the main objective of the intermediate levels is to identify and advance multiple potential important regions rather than to directly estimate the final failure probability. At this stage, the relative contribution of each ISC to the final failure probability has not yet been determined. Therefore, an equally weighted mixture is adopted to prevent centers with large original PDF values from dominating the sampling process prematurely, thereby maintaining the exploration capability for low-probability potential failure regions.

Then draw the $k$-th-level importance samples from the density in Eq.(16). The detailed sampling procedure is as follows. First, draw $\left\lceil \frac{N_k^*}{N_{\mathrm{ISC}_K}} \right\rceil$ samples following $h^k_{\boldsymbol{x}^k_{\mathrm{ISC},i}}(\boldsymbol{x})$. Traverse all ISCs to obtain $N_k = \left\lceil \frac{N_k^*}{N_{\mathrm{ISC}_K}} \right\rceil \times N_{\mathrm{ISC}_K}$ importance samples $\{\boldsymbol{x}_j^k\}_{j=1}^{N_k}$ and calculate their corresponding performance-function values $\{g(\boldsymbol{x}_j^k)\}_{j=1}^{N_k}$, where $N_k^*$ denotes the preset sample size at level $k$ and $N_k$ denotes the actual sample size generated because of the rounding operation.

9. **Extraction of conditional failure samples:** According to the performance-function values $\{g(\boldsymbol{x}_j^k)\}_{j=1}^{N_k}$, identify the $M_k$ sample points $\{\boldsymbol{x}_j^k\}_{j=1}^{M_k}$ falling into $F_{k-1}$, and denote their corresponding performance-function values by $\{g(\boldsymbol{x}_j^k)\}_{j=1}^{M_k}$.

10. **Update of the intermediate failure threshold:** Sort the above $M_k$ response values in descending order and denote the sorted results as $g(\boldsymbol{x}_{[1]}^k) > g(\boldsymbol{x}_{[2]}^k) > \cdots > g(\boldsymbol{x}_{[M_k]}^k)$. The $\lceil (1-p_0)M_k \rceil$-th response value is taken as the critical value $b_k$ of the intermediate failure event $F_k = \{\boldsymbol{x} : g(\boldsymbol{x}) \le b_k\}$, and $b_k = g(\boldsymbol{x}^k_{[\lceil (1-p_0)M_k \rceil]})$.

11. **Proceed to the next-level search:** Return to Step 3 for the next-level search.

*Stage 2: Construction of the final IS-PDF*

12. **Failure-ISC optimization and ISC correction:** Taking the ISCs $\{\boldsymbol{x}^m_{ISC,i}\}_{i=1}^{N_{KF}}$ screened from the failure samples at level $m$ as the initial iteration points, solve the following constrained optimization problem in the standard normal space using a local optimization method:

$$\min_{\boldsymbol{x}} \|\boldsymbol{x}\|, \quad s.t.\ g(\boldsymbol{x}) = 0 \tag{17}$$

This optimization process corrects each ISC onto the limit-state surface and moves ISC toward a position with a higher original probability density. This improves its representativeness as a final ISC. The optimized ISC set is denoted by $\{\boldsymbol{x}^{m,*}_{ISC,i}\}_{i=1}^{N_{KF}}$.

13. **Construction of the final-level IS-PDF and generation of importance samples:** First, the PDF value $\{\phi_n(\boldsymbol{x}^{m,*}_{ISC,i})\}_{i=1}^{N_{KF}}$ of each optimized ISC under the standard normal distribution is calculated, and the normalized weights $\gamma_i, i = 1,2,...,N_{KF}$ of all optimized ISCs are obtained from Eq. (18).

$$\gamma_i = \frac{\phi_n(\boldsymbol{x}^{m,*}_{ISC,i})}{\sum_{j=1}^{N_{KF}} \phi_n(\boldsymbol{x}^{m,*}_{ISC,j})}, i = 1,2,...,N_{KF} \tag{18}$$

Then, for each optimized ISC $\boldsymbol{x}^{m,*}_{\mathrm{ISC},i}$, construct a Gaussian component $h^m_{\boldsymbol{x}^{m,*}_{\mathrm{ISC},i}}(\boldsymbol{x})$ using $\boldsymbol{x}^{m,*}_{\mathrm{ISC},i}$ as the ISC and the identity covariance matrix as the covariance matrix. Based on these components, construct the final-level Gaussian-mixture IS-PDF.

$$h^m_{\boldsymbol{X}}(\boldsymbol{x}) = \sum_{i=1}^{N_{KF}} \gamma_i h^m_{\boldsymbol{x}^{m,*}_{\mathrm{ISC},i}}(\boldsymbol{x}) \tag{19}$$

Samples are then drawn from this density. The specific sampling process is as follows. According to the normalized weights $\{\gamma_i\}_{i=1}^{N_{KF}}$, the index of a Gaussian component is first randomly selected from the discrete distribution $Categorical(\gamma_1,...,\gamma_{N_{KF}})$. A sample is then generated from the corresponding Gaussian component.

Repeating this process $N_m$ times yields $N_m$ independent samples following the mixture density $h_{\boldsymbol{X}}^m(\boldsymbol{x})$.

It is worth noting in this step that mixture weights proportional to their original PDF values is used in this step to construct the final IS-PDF. This is because that the optimized failure ISCs correspond to failure-boundary locations with relatively high probability density in different local failure regions. This strategy can improve the ability of the final IS-PDF to cover the main regions contributing to the failure probability.

*Stage 3: Failure probability estimation*

14. **Final failure probability estimation:** Based on the IS samples $\{\boldsymbol{x}_j^m\}_{j=1}^{N_m}$ obtained at level *m*, estimate the final failure probability using Eq. (21):

$$\hat{P}_f = \frac{1}{N_m}\sum_{j=1}^{N_m} I_F(\boldsymbol{x}_j^m)\frac{\phi_n(\boldsymbol{x}_j^m)}{h_{\boldsymbol{X}}^m(\boldsymbol{x}_j^m)} \tag{20}$$

In summary, the proposed method first enlarges the variance of the initial-level IS-PDF, enhancing the ability of early samples to explore potential failure regions far from the original high-probability area. This provides rich global exploration information for subsequent stratified searching. K-means clustering then partitions the conditional samples inside intermediate failure events into multiple candidate regions with distinct spatial distributions, overcoming the limitation of a single ISC in representing multimodal or widely separated failure regions. Within each cluster, a dual-domain ISC screening mechanism is applied: for safe samples, the one with a smaller original PDF value is chosen as the ISC to push next-level samples farther away from the high-probability region; for failure samples, the one with a larger original PDF value is retained to preserve failure information near the high-density region. By combining variance enlargement, clustering-based identification, and safe-domain ISC extrapolation, samples advance level by level toward the limit-state surface along multiple promising directions, enabling rapid approximation of the failure boundary. Furthermore, the algorithm checks whether failure samples appear in all clusters to determine if all potential important failure regions have been sufficiently explored, thereby deciding when to terminate the intermediate-level progression.

In the final IS-PDF construction stage, the screened failure ISCs serve as starting points for local optimization on the limit-state surface, shifting them to failure-boundary positions with relatively high probability density within their respective local failure regions. Mixture weights are then assigned based on the original PDF values of these optimized ISCs, yielding a weighted Gaussian-mixture IS-PDF. This procedure preserves information from multiple dominant failure regions while improving the quality of each individual ISC, ultimately constructing a high-quality final IS-PDF that enhances both the efficiency and robustness of rare-event failure probability estimation for complex systems.

## 3. Examples

This section uses three examples to test the accuracy and efficiency of the proposed method. The three examples examine, respectively, a complex failure-domain problem that is difficult for the classical SuS method, a linear high dimensional problem with extremely small failure probability, and a nonlinear high dimensional problem with extremely small failure probability. The proposed method is denoted as AS-IS (adaptive stratified importance sampling).

In the three test examples for failure probability estimation, the results obtained by MCS with a large sample size are used as reference solutions. The proposed method is compared with existing methods for high dimensional small failure probability problems, including SuS-aCS (Papaioannou et al., 2015), SIS (Papaioannou et al., 2014), and SDIS (Cheng et al., 2023). Five evaluation metrics are used for the comparison:

the mean failure probability $\bar{P}_f$ over 30 independent runs, the coefficient of variation $CoV$ of the 30 failure probability estimates, the average relative error $\bar{\varepsilon}$ of the 30 independent failure probability estimates with respect to the reference solution, the relative error $\varepsilon_{\bar{P}_f}$ of the mean failure probability with respect to the reference solution, and the average number of performance-function calls *Ncall*. It should be noted that the comparison methods are selected according to the purpose of each example and the applicability conditions of the methods. In Example 3, because the failure probability decreases as the standard deviation of the input variables increases, SDIS, which constructs auxiliary failure probability sequences by variance enlargement, is not suitable and is therefore not included in the comparison for this example.

To ensure reproducibility of the comparison, the parameters of the comparison methods are set according to their corresponding references or the recommended settings in the publicly available source codes. Specifically, SuS-aCS uses 2000 conditional samples at each level, SIS uses 2000 IS samples at each level, and the SDIS parameters are set to NF = 100 and LEN = 5. For the proposed method, the sample size is selected so that the average number of performance-function calls is as close as possible to that of the most efficient comparison methods. This enables comparing the accuracy and efficiency of different methods under the similar computational cost.

*Example 1*

Example 1 selects the fourth representative example in (Breitung, 2019), which is difficult for the SuS method to solve effectively. This example mainly reflects the situation in which the main failure domains are blocked by high-performance-function regions and the topology of the failure domain changes during the level-by-level approximation process. The performance function can be expressed as follows.

$$g(X_1, X_2) = \frac{30}{\left(\frac{4(X_1+2)^2}{9} + \frac{X_2^2}{25}\right)^2 + 1} + \frac{20}{\left(\frac{(X_1-2.5)^2}{4} + \frac{(X_2-0.5)^2}{25}\right)^2 + 1} - 3 \tag{21}$$

As shown in Figure 1 (a), the performance function in this example has relatively complex geometric characteristics. At the same time, Figure 1 (b) shows that the contour shapes corresponding to $g(\boldsymbol{X})=15$ and $g(\boldsymbol{X})=0$ are clearly different, indicating that the geometry and connectivity of the conditional failure domain change as the intermediate failure threshold decreases level by level.

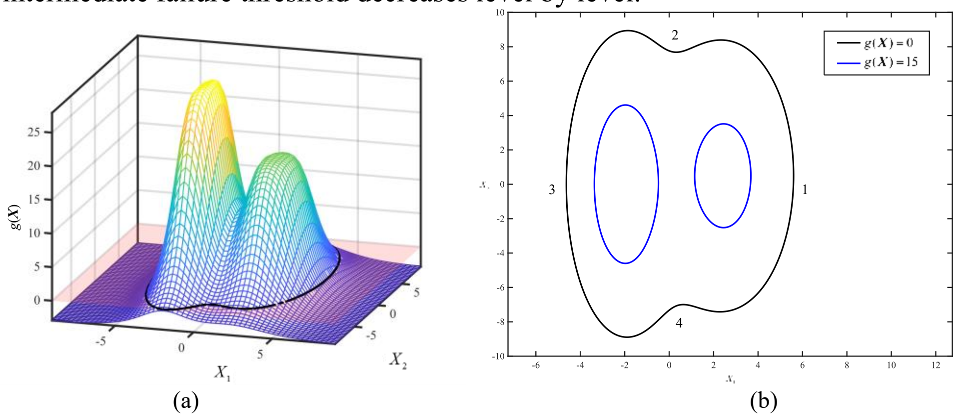


Figure 1 Performance surface and contours of $g(\boldsymbol{X})=15$ and $g(\boldsymbol{X})=0$ for Example 1.

During this process, sequential simulation samples tend to advance along locally accessible directions of decreasing function values near the original sampling center and preferentially cluster toward failure domains 2 and 4 in Figure 1 (b). And failure domains 1 and 3 contribute more to the final failure probability. Therefore, if samples fail to cross the local high-performance-function regions and enter these two domains, the dominant failure regions may not be adequately explored, leading to a substantial underestimation of the final failure probability.

Table 1 Failure probability estimates obtained by different methods for the problem in Example 1

| | $\bar{P}_f$ | $CoV$ | $\bar{\varepsilon}$ | $\varepsilon_{\bar{P}_f}$ | $Ncall$ |
|---|---|---|---|---|---|
| MCS | $2.015\times10^{-6}$ | 0.0498 | / | / | $2\times10^{8}$ |
| SuS-aCS | $2.287\times10^{-6}$ | 1.9533 | 1.5485 | 0.1349 | 23400 |
| SIS | $1.938\times10^{-6}$ | 2.395 | 1.500 | 0.038 | 19133 |
| SDIS | $2.189\times10^{-6}$ | 0.1610 | 0.1538 | 0.0863 | 3229.2 |
| AS-IS | $2.031\times10^{-6}$ | 0.0797 | 0.0611 | 0.0081 | 3222.4 |

Table 1 and Figure 2 and Figure 3 compare the performance of SuS-aCS, SIS, and SDIS in Example 1. SuS-aCS yields a mean failure-probability estimate of $2.287\times10^{-6}$, but its relative error and $CoV$ reach 0.1349 and 1.9533, respectively. As illustrated in Figure 2, its level-by-level samples tend to cluster in failure domains 2 and 4 and fail to adequately reach the dominant domains 1 and 3, leading to insufficient coverage and unstable estimates. SIS has a mean estimate of $1.938\times10^{-6}$, but its $CoV$ remains as high as 2.395 and its average relative error is 1.500. Figure 3 shows that its final IS-PDF is likewise concentrated mainly in domains 2 and 4, indicating that sequential IS-PDF updating does not necessarily overcome the omission of important regions caused by performance-function barriers and topological changes. By comparison, SDIS provides a mean estimate of $2.189\times10^{-6}$, with a $CoV$ of 0.1610 and an average of 3229.2 performance-function calls. Variance enlargement and directional sampling reduce the path dependence of local advancement; however, because its important directions are determined from randomly selected initial auxiliary failure samples, the coverage of the left and right dominant domains may still vary across runs, leaving its $CoV$ higher than that of AS-IS.

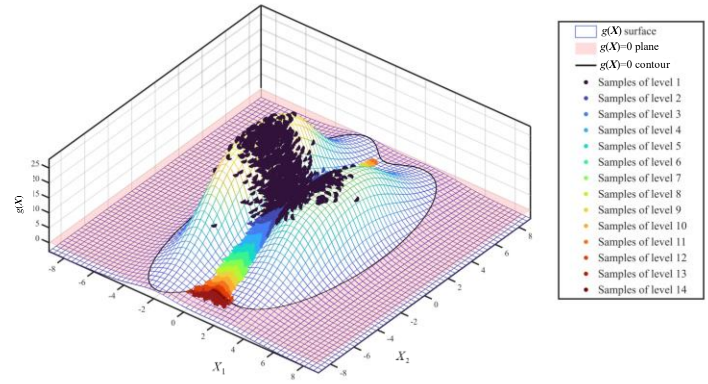


Figure 2 Sampling process of SuS-aCS method in the estimation of failure probability for the problem in Example 1

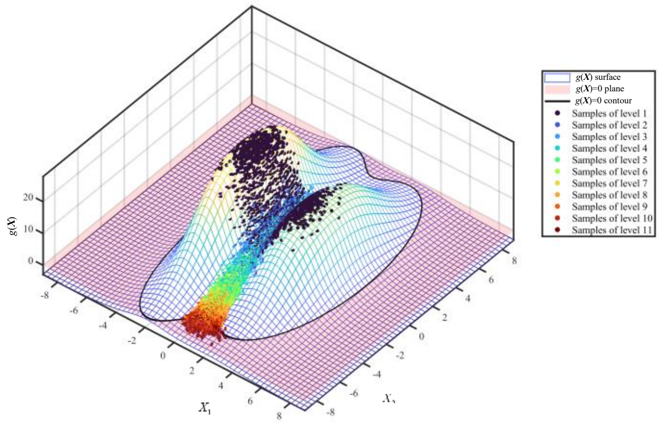


Figure 3 Sampling process of SIS method in the estimation of failure probability for the problem in Example 1

AS-IS exhibits better overall performance in this example. Its mean failure probability is $2.031\times10^{-6}$, which is highly consistent with the MCS reference value of $2.015\times10^{-6}$. The relative error of the mean is only 0.0081, and $CoV$ is 0.0797, which is clearly lower than those of SuS-aCS, SIS, and SDIS. Its average number of performance-function calls is 3222.4, almost the same as the 3229.2 calls required by SDIS, but its estimation stability is higher. These results indicate that, under a similar or lower computational cost, AS-IS can more stably identify and cover the dominant failure regions on both sides and thus construct a more appropriate final IS-PDF.

Figure 4 illustrates the mechanism: variance enlargement supports early exploration, K-means identifies candidate regions, and local optimization moves screened ISCs toward the dominant failure domains. The resulting final IS-PDF covers the principal contributors simultaneously.

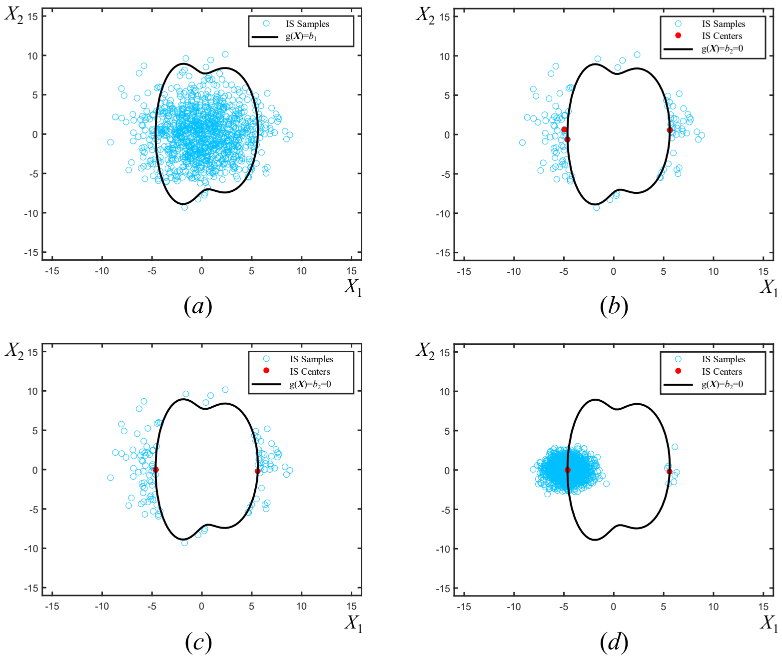


Figure 4 Sampling process of AS-IS method in the estimation of failure probability for the problem in Example 1

Therefore, the results of Example 1 show that when high-performance-function regions of the performance function obstruct the path between the original sampling center and important failure domains, the samples of SuS-aCS and SIS tend to extrapolate along locally accessible descent paths and consequently concentrate on non-dominant failure regions, while failing to sufficiently cover the truly important failure domains on the left and right sides. SDIS can alleviate this problem to some extent through variance enlargement and directional sampling, but its directional coverage is still affected by the randomness of the initial auxiliary failure samples. In contrast, AS-IS identifies potential important regions through K-means clustering and combines ISC optimization with multi-center Gaussian-mixture IS-PDF construction, enabling the ISCs to approach different dominant failure regions more stably and effectively. Therefore, AS-IS achieves the smallest relative error of the mean, a low *CoV*, and a number of performance-function calls comparable to that of SDIS in Example 1, demonstrating its advantages in accuracy, stability, and efficiency for problems with complex topological changes.

*Example 2 A high dimensional linear example with an extremely small failure probability*

Example 2 is mainly used to test the applicability of the proposed method to high dimensional problems with extremely small failure probability. For this purpose, the following high dimensional linear performance function is constructed:

$$g(\boldsymbol{X}) = 8 - \frac{1}{\sqrt{1000}} \sum_{i=1}^{1000} X_i \tag{22}$$

All random input variables follow standard normal distributions, and the theoretical solution of the failure probability is $P_f = \Phi(-8) \approx 6.221 \times 10^{-16}$. This example compares the proposed method with SuS-aCS, SIS, and SDIS. The solution results of different methods for Example 2 are shown in Table 2.

Table 2 Failure probability estimates obtained by different methods for the problem in Example 2

| | $\bar{P}_f$ | $CoV$ | $\bar{\varepsilon}$ | $\varepsilon_{\bar{P}_f}$ | $N_{call}$ |
|---|---|---|---|---|---|
| SuS-aCS | $6.761\times10^{-16}$ | 1.1487 | 0.7309 | 0.0869 | $3.373\times10^{4}$ |
| SIS | $4.343\times10^{-17}$ | 1.9417 | 0.9302 | 0.9302 | $3.103\times10^{4}$ |
| SDIS | $7.445\times10^{-16}$ | 1.9629 | 1.4588 | 0.1968 | $4.296\times10^{4}$ |
| AS-IS | $6.235\times10^{-16}$ | 0.0499 | 0.0415 | 0.0023 | $3.041\times10^{4}$ |

As shown in Table 2, the estimated mean of AS-IS is $6.235\times10^{-16}$, which is highly consistent with the theoretical solution, and the relative error of the mean is only 0.0023. Meanwhile, its *CoV* is 0.0499 and the average relative error is 0.0415, indicating that the proposed method can maintain high estimation accuracy and good single-run stability in this high dimensional extremely small failure probability problem.

Overall, in this example, AS-IS outperforms SuS-aCS, SIS, and SDIS in estimation accuracy, single-run stability, and computational efficiency. This demonstrates that the proposed method can more stably construct a high-quality IS-PDF with sufficient coverage of the dominant failure region, thereby improving the accuracy and stability of failure probability estimation.

*Example 3 A high dimensional nonlinear example with an extremely small failure probability*

Example 3 is mainly used to test the applicability of the proposed method to high dimensional nonlinear performance functions. For this purpose, the following test performance function is constructed:

$$g(\boldsymbol{X}) = -760 + \sum_{i=1}^{100} X_i + \sum_{i=1}^{100} X_i^2 - \sum_{i=1}^{100} X_i^3$$

The random variables follow normal distributions with a mean of -2 and a standard deviation of 0.4, $X_i \sim N(-2, 0.4^2), i = 1, 2, ..., 100$ . In the actual calculation, these non-standard normal variables are first standardized into independent standard normal variables, and the proposed algorithm is then implemented in the standard normal space.

Example 3 uses the MCS estimate as the reference solution and compares the proposed AS-IS method with SuS-aCS and SIS. Table 3 presents the solution results of different methods for Example 3.

Table 3 Failure probability estimates obtained by different methods for the problem in Example 3

| | $\bar{P}_f$ | $CoV$ | $\bar{\varepsilon}$ | $\varepsilon_{\bar{P}_f}$ | $N_{call}$ |
|---|---|---|---|---|---|
| MCS | $3.681\times10^{-10}$ | 0.05 | / | / | $1.0868\times10^{12}$ |
| SuS-aCS | $4.135\times10^{-10}$ | 0.3153 | 0.2754 | 0.1235 | $4.4\times10^{4}$ |
| SIS | $4.341\times10^{-10}$ | 1.0705 | 0.8461 | 0.1795 | $4.5\times10^{4}$ |
| AS-IS | $3.675\times10^{-10}$ | 0.0463 | 0.0368 | 0.0014 | $4.2267\times10^{4}$ |

The third example mainly verifies the capability of the proposed method in handling high dimensional nonlinear performance functions. As shown in Table 3, the estimated mean of AS-IS is $3.675\times10^{-10}$, which is highly consistent with the MCS reference solution, and the relative error of the mean is only 0.0014. Meanwhile, its $CoV$ is 0.0463 and the average relative error is 0.0368, both of which are the lowest among the three compared methods. The number of performance-function calls is only $4.2267\times10^{4}$, also the lowest among the three methods. These results indicate that the proposed method maintains high estimation accuracy while achieving good single-run stability and high computational efficiency.

## 4. Conclusion

This paper proposes a practical stratified importance sampling method to address the limitations of traditional stratified simulation methods in high dimensional small failure probability estimation, including sample correlation, insufficient identification of important failure regions, and limited construction quality of the final IS-PDF.

First, starting from the failure probability expression of SS-IS, this paper shows that when the final failure probability is evaluated by IS based on the final-level IS-PDF, the intermediate conditional failure probabilities no longer enter the final failure probability expression directly in a multiplicative form. Therefore, the primary role of the intermediate levels is not to accurately estimate the conditional failure probabilities level by level, but rather to progressively approach the limit-state surface, identify potential important failure regions, and provide effective information for constructing a high-quality final IS-PDF through sample screening, region advancement, and ISCs updating.

Based on this understanding, this paper constructs an adaptive stratified importance sampling strategy-AS-IS consisting of stratified search, ISC correction, and final-level IS estimation. The method first enlarges the sampling variance of the first-level IS-PDF to improve the initial exploration capability for potential important failure regions. It then uses K-means clustering to identify multiple candidate regions and screens safe ISCs and failure ISCs within each cluster. After failure samples are obtained in all current candidate regions, the failure ISCs are used as initial points and corrected by a local optimization method to positions on the limit-state surface with relatively high original probability density. The final-level Gaussian-mixture IS-PDF is then constructed based on the optimized ISCs, and the failure probability estimation under high dimensional conditions is completed.

The three examples show that AS-IS more reliably covers dominant failure regions and achieves accurate, stable estimates at competitive computational cost. Future work will combine the final-level IS-PDF construction with surrogate models for more complex high-dimensional applications.

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