Fuzzy Fields based on the Parallelepiped Dependency Model

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Abstract. In order to realistically characterize data uncertainty in engineering – and consequently to reliably assess safety – it is required to thoroughly investigate spatial and temporal dependencies of uncertainty, e.g. in material characteristics, geometrical or environmental parameters. Recognizing the fact that probability-based approaches are inappropriate for databases with a severe lack of knowledge, purely epistemic uncertainty concepts for modeling dependent quantities are required.

Several approaches to convex dependency modeling of interval quantities, as well as proposals for spatially dependent interval/fuzzy uncertainty modeling, exist. This contribution continues these researches by investigation of an existing fuzzy field concept. Based on discussion of limitations in the existing concept, novel developments in modeling and uncertainty quantification of fuzzy fields are presented. The developments are based on extending an existing interval dependency method, where multidimensional parallelepipeds characterize the underlying interval dependency.

Therefore, a dependency model is formulated by extension from interval to fuzzy fields. The proposed (auto-)interaction concept is in analogy to the (auto-)correlation concept of random fields. It is demonstrated how an eigenvalue decomposition of the auto-interaction structure with reduced orthogonal basis variables can be implemented to reduce the high-dimensional α-level optimization problem in uncertainty quantification. Finally, the novel methods are validated by uncertainty quantification examples in benchmark functions to demonstrate the safety assessment capabilities.

Keywords: polymorphic uncertainty quantification, possibility, fuzzy fields, structural analysis

1. Introduction

Both, purely aleatoric and polymorphic uncertainty are widely applied for spatial variability in engineering safety assessment. For instance, fields are defined by probabiltiy-boxes in (Do et al., 2016). Interval- and/or fuzzy-parametrized random fields constitute of a broad range of polymorphic uncertain field methods, such as in (Schietzold et al., 2019) and their research for challenging structural engineering problems is recently ongoing, see (Fina et al., 2023; Schietzold et al., 2024).

Material characteristics, for instance of timber as an inhomogeneous fiber composite, particularly require consideration of spatial uncertainty for engineering systems. Non-destructive spatial tests are increasingly affordable. Ultimately, local variations significantly determine the mechanical behavior of timber, and localization effects can cause component failure. However, data is always limited, only representing fractions of reality.

Additionally, besides natural variability, incompleteness remains. Therefore, concepts of possibility-based spatial uncertainty models are required, such as the interval field proposed by (Faes and Moens, 2017), which recognize the fact that probability-based approaches are inappropriate for databases with a severe lack of knowledge.

As this contribution is dedicated to uncertainty under severe lack of knowledge, purely epistemic models are in focus. Therefore, intervals and fuzzy quantities are defined in Section 2.1 as descriptions of uncertainty. As detailed in Section 2.1, a fuzzy analysis can be interpreted as multiple nested interval analyses in α-level optimizations. Hence, methods of interval uncertainty can be generalized to fuzzy uncertainty. Specifically, generalization to fuzzy fields is achieved by nesting of interval fields and, vice versa, interval fields can be defined as special case of fuzzy field approaches.

Orthogonal series expansion concepts are constitutive for many definitions of (polymorphic uncertain) random fields, see e.g. (Schietzold et al., 2019). This allows choosing a reduced number of orthogonal bases, which results in reduced numerical effort, see (Sudret and Der Kiureghian, 2000). For both, interval and fuzzy fields, documented methods are also based on orthogonal series expansion. According to (Luo et al., 2019), multi-dimensional ellipsoid convex models can be used to define such interval fields. In (Faes and Moens, 2017; Faes and Moens, 2019), interval fields are defined by an inverse distance weighting scheme and constituted by so-called control points in the observation domain. Contrarily, the concept of fuzzy fields in (Götz et al., 2016; Götz, 2017) is constituted by a distance-governed auto-dependency function, greatly inspired by the auto-correlation functions of (polymorphic uncertain) random field approaches. This method is briefly explained in Section 2.2, along with discussion of limitations and suggestions for improvement of the method. All these methods are based on different definitions of the underlying (auto-)dependencies within the fuzzy and interval fields. *Correlation* is only defined for random variables. In contrast, approaches defining dependencies of interval/fuzzy quantities are denoted as *interaction* methods.

According to (Jiang et al., 2015b), multi-dimensional ellipsoid convex models either heavily overestimate the intervals (ellipsoid encloses the hyperrectangle) or underestimate the corners of the interval-hypercube (hyperrectangle enclosing the ellipsoid). As an alternative, multi-dimensional parallelepiped (MP) convex models are suggested. This contribution's focus is on fuzzy field concepts constituted by auto-dependency functions and on interval fields as a special case of them. Consequently, a novel fuzzy field method is proposed in Section 2.4, based on a modification of the MP interval dependency model of (Jiang et al., 2015a), described in Section 2.3.

2. Theoretical Basis

2.1. Basic Uncertainty Models and Uncertainty Quantification

Uncertainty Characteristics

According to (Beer et al., 2013; Pannier et al., 2013), uncertainty can generally be distinguished in two characteristics: aleatoric and epistemic. Aleatoric uncertainty is due to natural variability and, therefore, not reducible. Contrarily, epistemic uncertainty describes all uncertainties that are reducible in theory, such as incompleteness, lack-of-knowledge and impreciseness.

This contribution follows the concept of (Graf et al., 2015), according to which the aleatoric uncertainty is mathematically modeled by probability-based methods, by stochastic quantities.

Furthermore, epistemic uncertainty is modeled by possibility theory with intervals or fuzzy quantities, as defined under the subsequent headlines. Moreover, in realistic engineering applications, a manifold of databases and corresponding uncertainty characteristics exist and, therefore, methods incorporating both, epistemic and aleatoric uncertainty are required. This combination of both characteristics is introduced as polymorphic uncertainty in (Graf et al., 2015) and much research in broad engineering fields is dedicated to it since then, for instance in (Faes and Moens, 2019; Schietzold et al., 2024; Harazin et al., 2026). However, the focus of this contribution is on problems under severe lack of knowledge in data and, hence, purely epistemic uncertainty models are investigated.

Intervals

An interval quantity $\mathbb{X}^{\mathcal{I}}$ is constituted by the mapping $\mu_{\mathbb{X}^{\mathcal{I}}} : \mathbb{R} \to \{0,\, 1\},\ x \mapsto \mu_{\mathbb{X}^{\mathcal{I}}}(x)$, with the interval membership function $\mu_{\mathbb{X}^{\mathcal{I}}}(x) = \{1 \text{ if } x \in \mathbb{X}^{\mathcal{I}}; 0 \text{ if } x \notin \mathbb{X}^{\mathcal{I}}\}$. Intervals, as subsets of real numbers, can be denoted by their left and right bounds as $[\mathbb{X}^{\mathcal{I}}_{\mathrm{L}},\, \mathbb{X}^{\mathcal{I}}_{\mathrm{U}}] = \{x \in \mathbb{R} \mid \mathbb{X}^{\mathcal{I}}_{\mathrm{L}} \leq x \leq \mathbb{X}^{\mathcal{I}}_{\mathrm{U}}\}$. Mirrored brackets, e.g. $]\mathbb{X}^{\mathcal{I}}_{\mathrm{L}},\, \mathbb{X}^{\mathcal{I}}_{\mathrm{U}}]$ or $]\mathbb{X}^{\mathcal{I}}_{\mathrm{L}},\, \mathbb{X}^{\mathcal{I}}_{\mathrm{U}}[$, indicate half-open or open intervals, respectively.

Besides the definition of intervals through bounds, in (Jiang et al., 2015b) the representation by centroid $\mathbb{X}^{\mathcal{I}}_{\mathrm{C}} = \mathrm{C}\,(\mathbb{X}^{\mathcal{I}}) = (\mathbb{X}^{\mathcal{I}}_{\mathrm{L}} + \mathbb{X}^{\mathcal{I}}_{\mathrm{U}})/2$ and radius $\mathbb{X}^{\mathcal{I}}_{\mathrm{R}} = \mathrm{R}\,(\mathbb{X}^{\mathcal{I}}) = (\mathbb{X}^{\mathcal{I}}_{\mathrm{U}} - \mathbb{X}^{\mathcal{I}}_{\mathrm{L}})/2$ is defined.

Interval analyses, as uncertainty quantification of system responses with interval input parameters, can be solved by interval arithmetic, but, according to (Hanss, 2005), are prone to overestimation due to the problem of interval dependency (wrapping effect). Numerical estimation of interval output can be achieved by solving the min. and max. optimization problems of computing the lower and upper bounds of the output interval(s) with respect to the interval input space.

The interval input space is defined by the Cartesian product $\mathcal{C}^{\mathcal{I}} = \mathbb{X}^{\mathcal{I}}_1 \times \cdots \times \mathbb{X}^{\mathcal{I}}_{n_{\mathcal{I}}}$ of all $n_{\mathcal{I}}$ quantities. It is interpreted as $n_{\mathcal{I}}$-dimensional hyperrectangle spanned by the interval quantities.

Fuzzy Quantities

A fuzzy quantity out of the fuzzy power set $\mathcal{F}(\mathbb{R})$ of all such fuzzy variables is denoted by $\mathbb{X}^{\mathcal{F}} \in \mathcal{F}(\mathbb{R})$ and constituted by the mapping $\mu_{\mathbb{X}^{\mathcal{F}}} : \mathbb{R} \to [0,\, 1],\ x \mapsto \mu_{\mathbb{X}^{\mathcal{F}}}(x)$, where $\mu_{\mathbb{X}^{\mathcal{F}}}$ is the fuzzy membership function. Following the definitions of (Zadeh, 1965), arbitrary functions can be formulated for the membership, but only convex fuzzy quantities are investigated in this contribution.

The extension principle is introduced in (Zadeh, 1965) as a fuzzy analysis, which is the quantification of the uncertainty of system responses due to fuzzy input parameters. In analogy to interval arithmetic, fuzzy arithmetic can be applied, see (Hanss, 2005), but the wrapping effect is also faced.

Fuzzy quantities are discretized by α-level cuts, each for a specific value $\alpha^* \in [0, 1]$, by

$$\mathbb{X}^{\mathcal{F}}_{\alpha=\alpha^*} = \{x \mid \mu_{\mathbb{X}^{\mathcal{F}}}(x) \geq \alpha^*\}, \tag{1}$$

see (Möller et al., 2000). This allows to reformulate the extension principle as a bunch of n_α interval analyses. Hence, fuzzy quantities can be interpreted as nested set of interval quantities and fuzzy system responses are estimated by the according interval analyses, the so called α-level-optimizations.

The Cartesian product $\mathcal{C}^{\mathcal{F}} = \mathbb{X}^{\mathcal{F}}_1 \times \cdots \times \mathbb{X}^{\mathcal{F}}_{n_{\mathcal{F}}}$ defines the joint of all $n_{\mathcal{F}}$ fuzzy quantities $\mathbb{X}^{\mathcal{F}}_i,\ i \in \{1, \ldots, n_{\mathcal{F}}\}$. Therefore, the joint membership function of the Cartesian product is defined

$$\mu_{\mathcal{C}^{\mathcal{F}}} : \qquad \mathbb{R}^{n_{\mathcal{F}}} \to [0,\, 1], \qquad (x_1, \ldots,\, x_{n_{\mathcal{F}}}) \mapsto \min\left(\mu_{\mathbb{X}^{\mathcal{F}}_1}(x_1), \ldots,\, \mu_{\mathbb{X}^{\mathcal{F}}_{n_{\mathcal{F}}}}(x_{n_{\mathcal{F}}})\right). \tag{2}$$

Investigations for efficient numerical fuzzy analysis methods with different approaches are documented, for instance α-level discretization-based or α-level free. Additionally, mixed approaches are recently proposed, such as the intermediately discretized extended α-level optimization in (Richter et al., 2025), enabling to combine the advantages of α-level-based and α-level-free approaches.

Convex fuzzy quantities with support $\mathbb{X}^{\mathcal{F}}_{\alpha=0} = [x_{\mathrm{l}},\, x_{\mathrm{r}}]$, deterministic core $\mathbb{X}^{\mathcal{F}}_{\alpha=1} = x_{\mathrm{m}}$, where $x_{\mathrm{l}} \leq x_{\mathrm{m}} \leq x_{\mathrm{r}}$, and with linear membership functions between $\mu_{\mathbb{X}^{\mathcal{F}}} = 0$ and $\mu_{\mathbb{X}^{\mathcal{F}}} = 1$ are called fuzzy triangular and their simplified notation is $\mathbb{X}^{\mathcal{F}} = \langle x_{\mathrm{l}},\, x_{\mathrm{m}},\, x_{\mathrm{r}} \rangle$, see (Möller and Beer, 2004).

2.2. Investigation and Discussion of an Existing Fuzzy Field Concept

Definition of the Method

In (Götz et al., 2016), a method for simulation of fuzzy fields with spatial auto-interaction between discrete observation points in three-dimensional domains is postulated, which is elaborated in (Götz, 2017). This method is further demonstrated in an n-dimensional approach, and advantages as well as limitations are investigated. Hence, a fuzzy field $\mathbb{X}^{\mathcal{FF}}$ is defined as the mapping $\mathbb{X}^{\mathcal{FF}} : \ \boldsymbol{\Theta} \rightarrow \mathcal{F}(\mathbb{R}), \quad \boldsymbol{\theta} \mapsto \mathbb{X}^{\mathcal{FF}}(\boldsymbol{\theta})$, which sets a fuzzy quantity of the powerset $\mathcal{F}(\mathbb{R})$ for each observation point θ. For a homogeneous fuzzy field, the same quantity $\mathbb{T}^{\mathcal{F}}_{\mathrm{target}}$ with field-constituting marginal membership $\mu_{\mathbb{T}^{\mathcal{F}}_{\mathrm{target}}}$ is defined at each θ in the observation domain $\boldsymbol{\Theta} \subset \mathbb{R}^{n_\Theta}$.

Within each realization of the discretized fuzzy field (each sample of the fuzzy analysis), auto-interactions should be realized between the n_θ observation points $\boldsymbol{\theta}_i,\ i \in 1, \dots, n_\theta$. As definition for these auto-interactions, an auto-interaction matrix $\mathbf{I}$ is postulated with the entries

$$\mathrm{I}_{ij} = \mathrm{i}_{\mathrm{kernel}} \left(\mathrm{d}_{\mathrm{metric}} \left(\boldsymbol{\theta}_i, \boldsymbol{\theta}_j \right) \right) \tag{3}$$

corresponding to all pairs $(\boldsymbol{\theta}_i, \boldsymbol{\theta}_j) \in \boldsymbol{\Theta} \times \boldsymbol{\Theta}$. It is constituted by an auto-interaction kernel function, which maps non-negative real values to interactions $\mathrm{i}_{\mathrm{kernel}} : \mathbb{R}^+ \rightarrow [-1,\, 1]$, based on the distance measure $\mathrm{d}_{\mathrm{metric}} : \boldsymbol{\Theta} \times \boldsymbol{\Theta} \rightarrow \mathbb{R}^+$. The kernel function is requested to fulfill $\mathrm{i}_{\mathrm{kernel}} \left(\mathrm{d}_{\mathrm{metric}} \left(\boldsymbol{\theta}_i,\, \boldsymbol{\theta}_i \right) \right) = \mathrm{i}_{\mathrm{kernel}}(0) = 1$ and, for physically reasonable uncertain fields, decays until $\mathrm{i}_{\mathrm{kernel}}(\infty) = 0$. The interaction matrix I is symmetric, since the distance measure is defined to be a metric.

An orthogonal series expansion is suggested in (Götz et al., 2016) to simulate auto-interaction, based on the decomposition of $\mathbf{I}$ into eigenvalues λ_i and eigenvectors $\boldsymbol{\phi}_i :\ i \in 1, \dots,\, n_\theta$. Hence, the fuzzy field, represented by a vector $\mathbf{X}^{\mathcal{FF}}$ of fuzzy quantities at the discrete observation points, is

$$\mathbf{X}^{\mathcal{FF}} = \sum_{i=1}^{n_\lambda} \mathbb{T}^{\mathcal{F}}_{\mathrm{scale},i} \cdot \sqrt{\lambda_i} \cdot \boldsymbol{\phi}_i. \tag{4}$$

Therefore, normalized fuzzy triangular coefficients $\mathbb{T}^{\mathcal{F}}_{\mathrm{scale},i} = \langle -1,\, 0,\, 1 \rangle$ are introduced for excitation of a (reduced) number of eigenmodes $n_\lambda \leq n_\theta$. The truncation of eigenmodes can reduce numerical effort, especially for $\lambda_i \approx 0$, where the low influence of the respective $\mathbb{T}^{\mathcal{F}}_{\mathrm{scale},i}$ is negligible in the fuzzy analysis. Truncation leads to reduced dimensionality of the fuzzy input space, see Eq. (2).

The further definitions are based on the α-level discretization concept, see Eq. (1). The vector of fuzzy quantities $\mathbf{X}^{\mathcal{FF}}$ from Eq. (4) is required to be transformed into the actually desired bounds of the investigated marginal membership function, constituted by the "target" fuzzy quantity $\mathbb{T}^{\mathcal{F}}_{\mathrm{target}}$.

In (Götz et al., 2016), an additional "scatter" interval quantity $\mathbb{T}^{\mathcal{I}}_{\text{scatter}} = [0,\, 1]$ is postulated, and the transformation is defined per α-level as scaling to the interval bounds by

$$\mathbf{X}^{\mathcal{FF}}_{\text{scaled},\alpha} = \mathbb{T}^{\mathcal{F}}_{\text{target},\alpha} + \mathbb{T}^{\mathcal{I}}_{\text{scatter}} \cdot \Big(\mathbf{U}^{\mathcal{FF}}_{\alpha} \cdot (b-a) + a\Big), \tag{5}$$

$$\text{with} \qquad a = \mathbb{T}^{\mathcal{F}}_{\text{target},\alpha,\text{L}} - \mathbb{T}^{\mathcal{F}}_{\text{target},\alpha}, \quad b = \mathbb{T}^{\mathcal{F}}_{\text{target},\alpha,\text{U}} - \mathbb{T}^{\mathcal{F}}_{\text{target},\alpha}, \tag{6}$$

$$\mathbf{U}^{\mathcal{FF}}_{\alpha} = \frac{\mathbf{X}^{\mathcal{FF}}_{\alpha} - \text{“min”}\big(\mathbf{X}^{\mathcal{FF}}_{\alpha}\big)}{\text{“max”}(\mathbf{X}^{\mathcal{FF}}_{\alpha}) - \text{“min”}(\mathbf{X}^{\mathcal{FF}}_{\alpha})}. \tag{7}$$

Therein, $\mathbb{T}^{\mathcal{F}}_{\text{target},\alpha}$ is the α-level cut of the field-constituting $\mathbb{T}^{\mathcal{F}}_{\text{target}}$. The equation components a and b represent the factors to scale to the marginal memberships lower and upper α-level bounds $\mathbb{T}^{\mathcal{F}}_{\text{target},\alpha,\text{L}}$ and $\mathbb{T}^{\mathcal{F}}_{\text{target},\alpha,\text{U}}$. Although, the definition of Eq. (6) is noted with swapped subtrahend and minuend in (Götz et al., 2016; Götz, 2017), the here presented notation is corrected to achieve the desired scaling, and it is verified by the existing FORTRAN code of (Götz, 2017).

With the separation of the component in Eq. (7), a slightly different notation is shown. By this, it is noted that $\mathbf{X}^{\mathcal{FF}}_{\alpha}$ is a vector of α-level cuts of the (unscaled) field representing vector of fuzzy quantities $\mathbf{X}^{\mathcal{FF}}$, hence, a vector of intervals. Accordingly, the min/max operators in Eq. (7) are quoted, since min/max are ambiguous as operators for a vector of intervals. However, the component $\mathbf{U}^{\mathcal{FF}}_{\alpha}$ in Eq. (5) is required to be the fuzzy field expansion of Eq. (4), but in a normalized form with bounds $[-1,\, 1]$ on each α-level. Based on these unit-hyperrectangles, the scaling to the bounds of the desired marginal membership is performed in Eq. (5). Therefore, the min/max operators in Eq. (7) are to be evaluated per entry such that in the vector, each entry is

$$\mathbb{U}^{\mathcal{FF}}_{\alpha,i} = \frac{\mathbb{X}^{\mathcal{FF}}_{\alpha,i} - \min\big(\mathbb{X}^{\mathcal{FF}}_{\alpha,i}\big)}{\max(\mathbb{X}^{\mathcal{FF}}_{\alpha,i}) - \min(\mathbb{X}^{\mathcal{FF}}_{\alpha,i})} = \frac{\mathbb{X}^{\mathcal{FF}}_{\alpha,i} - \mathbb{X}^{\mathcal{FF}}_{\alpha,\text{L},i}}{\mathbb{X}^{\mathcal{FF}}_{\alpha,\text{U},i} - \mathbb{X}^{\mathcal{FF}}_{\alpha,\text{L},i}}, \tag{8}$$

since min and max of an α-level in fact are its lower and upper bound, respectively.

An additional interval quantity $\mathbb{T}^{\mathcal{I}}_{\text{scatter}} = [0,\, 1]$ is introduced in Eq. (5) for scaling to the bounds of the marginal membership, so that corner points of the respective α-levels are reachable. All over all, the base quantities spanning the uncertain input space are $\mathbb{T}^{\mathcal{F}}_{\text{scale},i}$ with $i \in \{1, \ldots, n_\lambda\}$, where $n_\lambda \leq n_\theta$, $\mathbb{T}^{\mathcal{F}}_{\text{target},\alpha}$ and $\mathbb{T}^{\mathcal{I}}_{\text{scatter}}$, which results in an $(n_\lambda{+}2)$-dimensional hyperrectangle as interval input space of the optimization on each α-level for fuzzy analyses.

Discussion of the Method's Parameters

Since the method of (Götz et al., 2016) is formulated based on the α-level representation of fuzzy fields, its auto-interaction can also be interpreted as interval interactions in discretized sense. Hence, the further discussion is also based on interaction characteristics of intervals in the hyperrectangles that result at the α-levels.

As stated above, a fuzzy analysis with the proposed fuzzy fields of (Götz et al., 2016) leads to $n_{\text{input}} = n_\lambda{+}2$ dimensions of the fuzzy input space, dependent on the $n_\lambda < n_\theta$ eigenmodes taken into account, see Eq. (4). Although, such simulated fuzzy fields are discretized at n_θ observation points, more uncertain base variables than observation points are required by the method.

The formulation with these additional base variables leads to increased numerical effort (high dimensional optimization space of α-level optimization), but also to the disadvantage that the sampling in the mapping on each α-level from base variables to the fuzzy field realization

$$(\mathbb{T}^{\mathcal{F}}_{\text{scale},1} \times \cdots \times \mathbb{T}^{\mathcal{F}}_{\text{scale},n_\theta} \times \mathbb{T}^{\mathcal{F}}_{\text{target},\alpha} \times \mathbb{T}^{\mathcal{I}}_{\text{scatter}}) \nrightarrow \mathbf{X}^{\mathcal{FF}}_{\text{scaled},\alpha} \tag{9}$$

is not bijective. The missing one-to-one correspondence means that the resulting objective function for optimization algorithms has duplicates for several combinations of design variables, which might be challenging for the global α-level optimization, according to (Blank and Deb, 2020).

Generally, the definitions of scale and scatter base variables $\mathbb{T}^{\mathcal{F}}_{\text{scale},i} = \langle -1,\, 0,\, 1\rangle$ and $\mathbb{T}^{\mathcal{I}}_{\text{scatter}} = [0,\, 1]$ in the formulation of Eq. (5) are questionable. Obviously, the $\mathbb{T}^{\mathcal{F}}_{\text{scale},i}$ as fuzzy triangular quantities lead to different expansions $\mathbf{X}^{\mathcal{FF}}_{\alpha}$ in Eq. (4) and further on to different unit-hyperrectangles $\mathbf{U}^{\mathcal{FF}}_{\alpha}$ for each α-level, where Eq. (8) becomes

$$\mathbb{U}^{\mathcal{FF}}_{\alpha,i} = \begin{cases} [0,1] & \text{as long as } \mathbb{X}^{\mathcal{FF}}_{\alpha,\text{U},i} \neq \mathbb{X}^{\mathcal{FF}}_{\alpha,\text{L},i}, \text{ hence, for } \alpha < 1, \text{ where } \mathrm{R}\,(\mathbb{T}^{\mathcal{F}}_{\text{scale},\alpha,i}) > 0 \text{ or} \\ \frac{0}{0} \Rightarrow 0 & \text{if all } \mathbb{X}^{\mathcal{FF}}_{\alpha=1,i} = [0,\, 0] = 0, \text{ hence for } \alpha = 1, \text{ where all } \mathbb{T}^{\mathcal{F}}_{\text{scale},\alpha=1,i} = 0, \\ & \text{which is the deterministic core of the fuzzy triangulars.} \end{cases} \tag{10}$$

Specifically, for $\alpha = 1$, Eq. (5) evaluates to

$$\begin{aligned} \mathbf{X}^{\mathcal{FF}}_{\text{scaled},\alpha=1} &= \mathbb{T}^{\mathcal{F}}_{\text{target},\alpha=1} + \mathbb{T}^{\mathcal{I}}_{\text{scatter}} \cdot \Big(\mathbf{U}^{\mathcal{FF}}_{\alpha=1} \cdot (b-a) + a\Big) \\ &= \mathbb{T}^{\mathcal{F}}_{\text{target},\alpha=1} + \mathbb{T}^{\mathcal{I}}_{\text{scatter}} \cdot \Big(\begin{bmatrix}0 & \cdots & 0\end{bmatrix}^{\intercal} + \mathbb{T}^{\mathcal{F}}_{\text{target},\alpha=1,\text{L}} - \mathbb{T}^{\mathcal{F}}_{\text{target},\alpha=1}\Big), \end{aligned} \tag{11}$$

which means that each realization of the fuzzy field has identical entries for all observation points, hence, realizations are constant in the observation domain. In other words, all realizations on $\alpha = 1$ are unexpectedly fully interacted, provided that the determination of $\frac{0}{0} \Rightarrow 0$ of Eq. (10) numerically is well handled in the implementation of the method at all. However, for all levels $\alpha < 1$, the interaction is defined as intended and the radii $\mathrm{R}\,(\mathbb{T}^{\mathcal{F}}_{\text{scale},\alpha<1,i})$ are irrelevant, since the fuzzy triangular quantity is symmetrically defined with all centroids $\mathrm{C}\,(\mathbb{T}^{\mathcal{F}}_{\text{scale},\alpha,i}) = 0$ and the resulting α-levels of the field expansion are all scaled to the unit-hyperrectangle by Eq. (8).

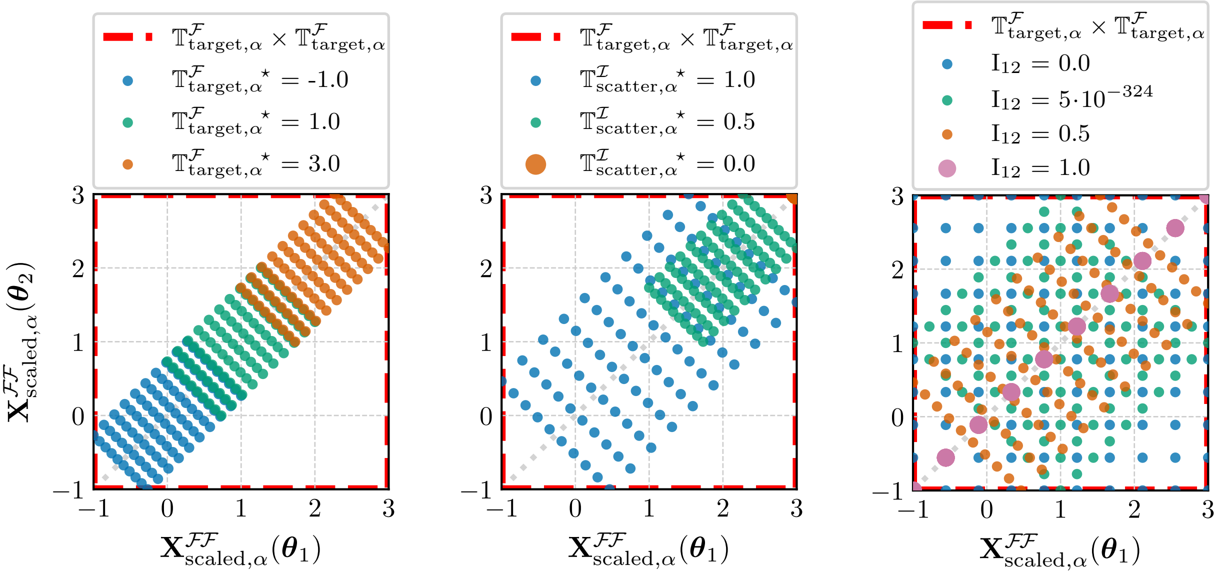
Due to identity of the two fuzzy quantities $\mathbb{T}^{\mathcal{F}}_{\text{target}}$ in Eq. (11), and recalling that $\mathbb{T}^{\mathcal{I}}_{\text{scatter}} = [0,\, 1]$,

$$\begin{aligned} \mathbf{X}^{\mathcal{FF}}_{\text{scaled},\alpha=1}\Big(\mathbb{T}^{\mathcal{I}}_{\text{scatter}}\Big) &\subseteq \mathbf{X}^{\mathcal{FF}}_{\text{scaled},\alpha=1}\Big({\mathbb{T}^{\mathcal{I}}_{\text{scatter}}}^{\star} = 0\Big) \\ &\subseteq \mathbb{T}^{\mathcal{F}}_{\text{target},\alpha=1} + 0 \cdot \left(\begin{bmatrix}0\\ \vdots \\ 0\end{bmatrix} + \mathbb{T}^{\mathcal{F}}_{\text{target},\alpha=1,\text{L}} - \mathbb{T}^{\mathcal{F}}_{\text{target},\alpha=1}\right) = \begin{bmatrix}\mathbb{T}^{\mathcal{F}}_{\text{target},\alpha=1} \\ \vdots \\ \mathbb{T}^{\mathcal{F}}_{\text{target},\alpha=1}\end{bmatrix} \end{aligned} \tag{12}$$

shows redundancy. Hence, the target fuzzy quantity $\mathbb{T}^{\mathcal{F}}_{\text{target},\alpha=1}$ at all observation points is already parametrized by $\mathbb{T}^{\mathcal{I}}_{\text{scatter}} = 0$ at $\alpha = 1$ and redundant realizations are simulated for $\mathbb{T}^{\mathcal{I}}_{\text{scatter}} =]0, 1]$.

Discussion of the Method's Interaction Characteristics

Finally, an unintentional characteristic of the method from (Götz et al., 2016) is investigated, leading to a discontinuity at transition from case of low interaction to the marginal case of no interaction.



a) $\mathbb{T}^{\mathcal{I}}_{\text{scatter},\alpha} = 0.5$; $\mathrm{I}_{12} = 0.5$ b) $\mathbb{T}^{\mathcal{F}}_{\text{target},\alpha} = 3.0$; $\mathrm{I}_{12} = 0.5$ c) $\mathbb{T}^{\mathcal{I}}_{\text{scatter},\alpha} = 1.0$; $\mathbb{T}^{\mathcal{F}}_{\text{target},\alpha} = 1.0$

Figure 1. Influence of the parameters on the auto-interaction between two observation points $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_2$ at a specific α-level, where $\mathbb{T}^{\mathcal{F}}_{\text{target},\alpha} = [-1, 3]$, with $\mathbb{T}^{\mathcal{F}}_{\text{scale},\alpha,i} = [-1, 1]$ for each $i \in \{1, 2\}$ discretized by 10 samples and all $n_\lambda = 2$ eigenmodes in Python implementation. Starred parameters $\bullet^{\star}$ indicate interval quantities fixed to deterministic values.

Therefore, the marginal membership functions at two discrete observation points $\mathbb{T}^{\mathcal{F}}_{\text{target}}(\boldsymbol{\theta}_1)$ and $\mathbb{T}^{\mathcal{F}}_{\text{target}}(\boldsymbol{\theta}_2)$ of a fuzzy field are considered. Fuzzy field realizations with interactions by Eq. (5) are restricted to a subset of the α-level cuts' Cartesian products $\mathcal{C}^{\mathcal{F}}_{\alpha} = \mathbb{T}^{\mathcal{F}}_{\text{target},\alpha}(\boldsymbol{\theta}_1) \times \mathbb{T}^{\mathcal{F}}_{\text{target},\alpha}(\boldsymbol{\theta}_1)$, as shown in Figure 1.

For $\mathrm{I}_{12} = 1$, all realizations of fully interacted intervals are expected to be on the diagonal spanned by $\left(\mathbb{T}^{\mathcal{F}}_{\text{target},\alpha,\text{L}}(\boldsymbol{\theta}_1), \mathbb{T}^{\mathcal{F}}_{\text{target},\alpha,\text{L}}(\boldsymbol{\theta}_2)\right)$ and $\left(\mathbb{T}^{\mathcal{F}}_{\text{target},\alpha,\text{R}}(\boldsymbol{\theta}_1), \mathbb{T}^{\mathcal{F}}_{\text{target},\alpha,\text{R}}(\boldsymbol{\theta}_2)\right)$. If $\mathrm{I}_{12} = 0$, the non-interacted intervals' sampling should reach every point in $\mathcal{C}^{\mathcal{F}}_{\alpha}$. These bounding cases are both enabled by the method and a validation is provided in Figure 1c). Additionally, it would be desired that the interaction models influence on the condition of the resulting α-level optimization problem is negligible for all subsets of the problem, but doubts about this are already noted in Eq. (9).

With Eq. (4), the interaction is expanded by normalized fuzzy quantities and a linear transformation is applied by Eq. (5) to match the bounds of the marginal membership function. Therefore, the desired interaction characteristics apply to arbitrary marginal membership functions.

In Figure 1, the influence of a fuzzy field's input quantities (1a and 1b) and of the interaction parameter (1c) on the auto-interaction is displayed. For each sample, Eq. (4) is evaluated with different parameters described in the respective subplot. The investigation of three parameter combinations for $\mathbb{T}^{\mathcal{F}}_{\text{target},\alpha}$, with fixed $\mathrm{I}_{12} = 0.5$ and $\mathbb{T}^{\mathcal{I}}_{\text{scatter},\alpha} = 0.5$, in Figure 1a), shows that several subsets of identical realizations, disadvantageous for optimization algorithms, exist. In Figure 1b), $\mathbb{T}^{\mathcal{F}}_{\text{target},\alpha} = 3.0$ is fixed, while the interaction is as in a) and the influence of $\mathbb{T}^{\mathcal{I}}_{\text{scatter},\alpha}$ is investigated.

It is shown that the interaction model relies on the width of $\mathbb{T}^{\mathcal{I}}_{\text{scatter},\alpha}$ to reach the diagonal corners of $\mathcal{C}^{\mathcal{F}}_{\alpha}$. However, this parameter also leads to the same problem of overlapping subsets

In Figure 1c), the influence of the auto-interaction parameter on the scaling of the subspace of the Cartesian product is shown. The cases $\mathrm{I}_{12} = 0$ and $\mathrm{I}_{12} = 1$ are expected, as noted before, and all points within $\mathcal{C}^{\mathcal{F}}_{\alpha}$ are reachable for them. Contrarily, for any $\mathrm{I}_{12} > 0$, the domain gets restricted. This is validated by the double precision machine epsilon > 0, as LAPACK routines are used for eigenvector computation, which is $\mathrm{I}_{12} = 5{\cdot}10^{-324}$, see Figure 1c). The restricted domain is defined by the rays parallel to the diagonal that connect the centroids $\mathbb{T}^{\mathcal{F}}_{\text{target},\alpha,\mathrm{C}}$ of the α-level of the marginal membership function. This undesired discontinuity is due to the orthogonal expansion of Eq. (4), which results in unit base vectors for the eigenvectors in case of $\mathbf{I} = \operatorname{diag}(1, \cdots, 1)$, the identity matrix. However, as soon as $\mathbf{I}_{ij} > 0$ the coordinate transformation, by 45° in Figure 1, takes place.

In result to this discontinuity, only fractions of the Cartesian product ($\frac{3}{4}$ area in two-dimensional case) are represented, limiting the applicability to realistic interactions. Especially, application for distance dependent auto-interactions as in Eq. (3) is questionable, since it is anticipated that $\lim_{\mathrm{d}(\boldsymbol{\theta}_i,\boldsymbol{\theta}_j)\to\infty}(I_{ij}) = 0$, and accordingly, far distanced observation points should be fully independent.

Conclusions and Improvement Suggestions for the Method

Based on observations in Eqs. (10)–(12), the following modifications are suggested:

a) To avoid resulting observation points unintentionally being fully auto-interacted at $\alpha = 1$, it is suggested to redefine the scale parameters by $\mathbb{T}^{\mathcal{I}\,\text{redef.}}_{\text{scale},i} := [-1, 1]$ as interval quantities. By this redefinition, it is ensured that the normalization to unit-hyperrectangle and scaling to the bounds of the marginal membership function are defined respectively for all α-levels.

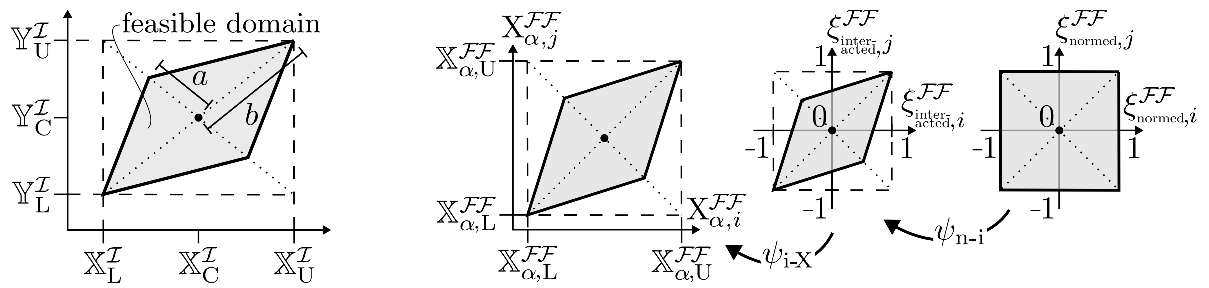
b) Contrarily, if constant realizations of the fuzzy field (fully interacted over all observation points) are actually intended as additional case, it is still suggested to redefine $\mathbb{T}^{\mathcal{I}\,\text{redef.}}_{\text{scale},i} := [-1, 1]$ as in a) and, additionally, $\mathbb{T}^{\mathcal{I}\,\text{fully-interacted}}_{\text{scale},i} := 0$ for all i and $\mathbb{T}^{\mathcal{I}\,\text{fully-interacted}}_{\text{scatter}} := 0$ can be set in a separate simulation, to enable an efficient fuzzy analysis without the redundancies of Eq. (12).

In conclusion, the numerical effort of over-parametrization with $n_{\text{input}} > n_\theta$ (all eigenmodes considered), the resulting ill-conditioning of the optimization problem and the non-physical discontinuity at transition case to zero interaction remain as disadvantages of the proposed method.

2.3. Dependency Model for Interval and Fuzzy Quantities – Interaction by a Multidimensional Parallelepiped

The multidimensional parallelepiped (MP) model for interval interaction is defined in several concepts by the same authors e.g. in (Jiang et al., 2015b) and specifically for interval arithmetic based uncertainty quantification formulated in (Jiang et al., 2015a). Therein, the MP model definition is founded on a dependency coefficient $\mathrm{i}^{\text{MP}}(\mathbb{X}^{\mathcal{I}}, \mathbb{Y}^{\mathcal{I}}) = (b-a)/(b+a)$, defined for the two-dimensional interaction between intervals $\mathbb{X}^{\mathcal{I}}$ and $\mathbb{Y}^{\mathcal{I}}$. Therein, the parameters a and b are defined by the parallelogram, describing the feasible area of interacted samples of the two intervals, and fixed to the diagonals as displayed in Fig. 2a). Based on this idea, the multidimensional interaction matrix $\mathbf{I}^{\text{MP}}$ for interactions between all intervals $\mathbb{X}_i$ and $\mathbb{X}_j$ is defined as symmetric matrix with entries

$$\mathrm{I}^{\text{MP}}_{ij} = \mathrm{I}^{\text{MP}}_{ji} = i^{\text{MP}}(\mathbb{X}^{\mathcal{I}}_i, \mathbb{X}^{\mathcal{I}}_j). \tag{13}$$



a) Parallelogram defining the dependency coefficient of the MP model in 2-dim.

b) Mapping scheme for α-level of MP fuzzy field – though it is noted that effective projections of n-dim. MPs are *not* parallelograms.

Figure 2. Multidimensional parallelepiped (MP) as interaction for interval interaction and fuzzy fields.

The MP for the n-dimensional interval interaction is defined by the feasible domain

$$\left| \mathbf{I}^{\mathrm{MP}\,-1}\,\mathbf{T}^{-1}\,\mathbf{R}^{-1}\left(\mathbf{X}^{\mathcal{I}} - \mathrm{C}\left([\mathbf{X}^{\mathcal{I}}]\right)\right) \right| \leq \mathbf{1}. \tag{14}$$

Therein, the norm denotes entry-wise absolute values, $\mathbf{1} = [1 \;\cdots\; 1]^{\intercal}$ is the all-ones vector and

$$\mathbf{T} = \mathrm{diag}\,(w_1, \ldots, w_n), \qquad w_i = 1/\sum_{j=1}^{n} |\mathrm{I}_{ij}|, \qquad \mathbf{R} = \mathrm{diag}\,(\mathrm{R}(\mathbb{X}_1^{\mathcal{I}}), \ldots, \mathrm{R}(\mathbb{X}_n^{\mathcal{I}})), \tag{15}$$

by applying the interval radius $\mathrm{R}(\bullet)$ and (entry-wise) centroid $\mathrm{C}([\bullet])$, see Section 2.1.

Although it is stated in (Jiang et al., 2015a) that the MP model is solely based on the bounds and mutual interactions of the intervals, it has to be noted that projections of an MP to the (i, j)-coordinate planes are not generally parallelograms, in contrast to the two-dimensional case, see Figure 2a). Moreover, in (Jiang et al., 2015a) an argumentation for the modeling of interaction coefficients for the multidimensional examples is omitted and in (Jiang et al., 2015b) data-based modeling of parallelograms in (i, j)-coordinate planes is shown, but the deviation to the geometry of the resulting parallelepiped is not quantified. Nevertheless, the idea of the MP is promising for characterization of multidimensional interval interactions and will be extended in Section 2.4, but the constituting assumption of a parallelogram with i^{MP} is not followed further.

2.4. Development of a Fuzzy Field Method by Modification and Extension of the Multidimensional Parallelepiped Interval Dependency

Based on the investigation of some non-negligible disadvantages in the existing fuzzy field approach, pointed out in Section 2.2, a novel definition of fuzzy fields is presented in this section. Therefore, it is recalled that fuzzy fields in α-level discretized representation can be interpreted as nested interval field models. Hence, the idea of an MP model of Section 2.3 is modified to an auto-interaction formulation for interval fields and, accordingly, extended to auto-interacted fuzzy fields.

Most importantly, it is shown at the end of this section that the modified MP model can be represented by an orthogonal series expansion, allowing for an eigenvalue decomposition that can be excited by a reduced number of orthogonal base variables. Hence, the numerical feasibility in fuzzy analyses of fuzzy fields of many observation points is enabled by the proposed method.

Definition of the Modification and Extension to Fuzzy Fields
For a sampling oriented definition of MP-based fuzzy fields, based on α-level discretization, the optimization design space of a discrete α-level can be interpreted as centered unit-hypercube $[-1,1]^{n_\theta}$, noted as vector of independent normed base intervals $\boldsymbol{\xi}^{\mathcal{FF}}_{\text{normed},\alpha} = ([-1,1] \ \cdots \ [-1,1])^\intercal$, with n_θ entries corresponding to the number of observation points in the fuzzy field. Their transformation to interacted intervals, forming an MP in analogy with Eq. (14), is defined by

$$\psi_{\text{n-i}}: \ \boldsymbol{\xi}^{\mathcal{FF}}_{\text{normed},\alpha} \mapsto \boldsymbol{\xi}^{\mathcal{FF}}_{\text{interacted},\alpha}, \quad \boldsymbol{\xi}^{\mathcal{FF}}_{\text{interacted},\alpha} = \mathcal{S}^{\pm}(\mathbf{I})\, \boldsymbol{\xi}^{\mathcal{FF}}_{\text{normed},\alpha}, \tag{16}$$

where $\mathbf{I}$ is the interaction matrix constituted by Eq. (3), characterizing the fuzzy field's distance-dependent auto-interaction. In (Jiang et al., 2015a), a normalized interaction matrix, denoted as shape matrix, is proposed for the transformation to the MP, by the row-wise normalization matrix of Eq. (15), to enforce the bounds of the centered unit-hypercube $[-1,1]^{n_\theta}$ for the transformed $\boldsymbol{\xi}^{\mathcal{FF}}_{\text{interacted},\alpha}$. This shape matrix is not necessarily symmetric, and therefore, an orthogonal basis spanned by its eigenvectors in $\mathbb{R}^{n_\theta}$ is not guaranteed. It is noted that this fact is not obvious in the examples of (Jiang et al., 2015a, Eqs. (26, 31, 38, 46, 49, 52, 56)), since only interaction matrices – mostly two-dimensional – with a priori identical row and column sums are presented therein.

In this contribution, an MP model modification is proposed by incorporating the Sinkhorn-Knopp algorithm, to achieve a doubly stochastic matrix form of the interaction matrix, see (Sinkhorn and Knopp, 1967). The algorithm is implemented in an adopted form to retain the entry signs in the matrix, since negative interactions are possible. By this, for Eq. (16), the component $\mathcal{S}^{\pm}(\mathbf{I})$ is referred to as the signed Sinkhorn form of the interaction matrix. It constitutes the mathematical precondition for the proper orthogonal decomposition, as shown at the end of this section.

As final step in the mapping chain, which in total is visualized in Figure 2b),

$$\psi_{\text{i-X}}: \ \boldsymbol{\xi}^{\mathcal{FF}}_{\text{interacted},\alpha} \mapsto \mathbf{X}^{\mathcal{FF}}_{\alpha}, \quad \mathbf{X}^{\mathcal{FF}}_{\alpha} = \mathbf{R}\, \boldsymbol{\xi}^{\mathcal{FF}}_{\text{interacted},\alpha} + \mathrm{C}([\mathbf{X}^{\mathcal{FF}}_{\alpha}]) \tag{17}$$

is defined, equivalent to Eq. (14), to scale the MP to the desired bounds of the marginal membership. The vector $\mathbf{X}^{\mathcal{FF}}_{\alpha}$ of auto-interacted intervals, corresponding to the observation points of the fuzzy field, is achieved. For a homogeneous fuzzy field, the same marginal membership function $\mu_{\mathbb{X}^{\mathcal{F}}}$, of the field-constituting fuzzy quantity $\mathbb{X}^{\mathcal{F}}$, is defined for all observation points, as in Section 2.2. Hence, the radii and centroids of all entries of $\mathbf{X}^{\mathcal{FF}}_{\alpha}$ are all identical with the field-constituting fuzzy quantity, with $\mathrm{R}\,(\mathrm{X}^{\mathcal{FF}}_{\alpha,i}) = \mathbb{X}^{\mathcal{F}}_{\mathrm{R}}$ and $\mathrm{C}\,(\mathrm{X}^{\mathcal{FF}}_{\alpha,i}) = \mathbb{X}^{\mathcal{F}}_{\mathrm{C}}$, respectively, for $i \in 1, \ldots, n_\theta$.

Orthogonal Decomposition and Reduced Basis Expansion
To enable numerically efficient fuzzy analyses, a reduced expansion is proposed for the MP-based fuzzy field method. Therefore, it is recalled that the interaction matrix is represented by its signed Sinkhorn form $\mathcal{S}^{\pm}(\mathbf{I})$ in Eq. (16). Eigenvectors of a Sinkhorn matrix are orthogonal (due to symmetry) and the largest (first if ordered) eigenvalue is $\lambda_1 = 1$. The same properties are also assumed to hold for the signed Sinkhorn form. Hence, the first eigenvector $\boldsymbol{\phi}_1$ of $\mathcal{S}^{\pm}(\mathbf{I})$ is oriented in direction of the largest diagonal of the multidimensional parallelepiped.

By all eigenvalues λ_i and eigenvectors $\boldsymbol{\phi}_i$ with $i \in 1, \ldots, n_\lambda$ of $\mathcal{S}^{\pm}(\mathbf{I})$ and $\boldsymbol{\Lambda} = \text{diag}\,(\lambda_1, \ldots, \lambda_{n_\lambda})$, the orthogonal expansion is

$$\mathcal{S}^{\pm}(\mathbf{I}) = \boldsymbol{\Phi}\ \boldsymbol{\Lambda}\ \boldsymbol{\Phi}^{\intercal}, \qquad \text{where} \quad \boldsymbol{\Phi} = \begin{bmatrix} | & & | \\ \boldsymbol{\phi}_1 & \cdots & \boldsymbol{\phi}_{n_\lambda} \\ | & & | \end{bmatrix}_{n_\lambda \times n_\theta}, \tag{18}$$

with respectively sorted eigenvalues $\lambda_i > \lambda_j : i < j$ and normed eigenvectors upright in the matrix.

Instead of the n_θ interval quantities in Eq. (16), it is desired to excite the eigenmodes by a reduced number n_λ of base intervals, with α-level notation in the vector $\boldsymbol{\zeta}_\alpha$, and postulated that

$$\psi_{\zeta\text{-i}}:\ \boldsymbol{\zeta}_\alpha \mapsto \boldsymbol{\xi}^{\mathcal{FF}}_{\text{interacted},\alpha}, \quad \boldsymbol{\xi}^{\mathcal{FF}}_{\text{interacted},\alpha} = \boldsymbol{\Phi}\ \boldsymbol{\Lambda}\ \boldsymbol{\zeta}_\alpha \quad \text{if } n_\lambda = n_\theta. \tag{19}$$

It is requested that this is an equivalent reducible basis orthogonal representation of the mapping $\psi_{\text{n-i}}$ to the multidimensional parallelepiped in the centered unit-hypercube, with

$$\psi_{\zeta\text{-i}} \stackrel{!}{=} \psi_{\text{n-i}} \quad \Leftrightarrow \quad \boldsymbol{\Phi}\ \boldsymbol{\Lambda}\ \boldsymbol{\zeta}_\alpha = \mathcal{S}^{\pm}(\mathbf{I})\, \boldsymbol{\xi}^{\mathcal{FF}}_{\text{normed},\alpha} \tag{20}$$

$$= \boldsymbol{\Phi}\ \boldsymbol{\Lambda}\ \boldsymbol{\Phi}^{\intercal}\ \boldsymbol{\xi}^{\mathcal{FF}}_{\text{normed},\alpha}, \qquad \text{by Eq. (18), and hence,}$$

$$\Leftrightarrow \boldsymbol{\zeta}_\alpha = \boldsymbol{\Phi}^{\intercal}\ \boldsymbol{\xi}^{\mathcal{FF}}_{\text{normed},\alpha}. \tag{21}$$

Accordingly with $n_\lambda < n_\theta$, an approximation of the fuzzy field is achieved by incorporating a reduced orthogonal basis in Eq. (19). The mapping, explicitly noting the matching matrix dimensions, reads

$$\psi_{\zeta\text{-i,red}}:\ \boldsymbol{\zeta}^{\text{red}}_\alpha \mapsto \boldsymbol{\xi}^{\mathcal{FF},\text{approx}}_{\text{interacted},\alpha}, \quad \left[\boldsymbol{\xi}^{\mathcal{FF},\text{approx}}_{\text{interacted},\alpha}\right]_{n_\theta} = \left[\boldsymbol{\Phi}^{\text{red}}\right]_{n_\theta \times n_\lambda}\ \left[\boldsymbol{\Lambda}^{\text{red}}\right]_{n_\lambda \times n_\lambda}\ \left[\boldsymbol{\zeta}^{\text{red}}_\alpha\right]_{n_\lambda}, \tag{22}$$

where $\boldsymbol{\Phi}^{\text{red}}$ and $\boldsymbol{\Lambda}^{\text{red}}$ are the reduced matrices of n_λ eigenvectors and eigenvalues, respectively. Therefore, the base intervals in $\boldsymbol{\zeta}^{\text{red}}$ define the design variables of the α-level optimization by a reduced n_θ-dimensional optimization problem. However, a challenge remains in the sampling of $\boldsymbol{\zeta}^{\text{red}}$, which requires to be compliant with Eq. (21). This equivalence is accounted for by incorporating the constraints of the centered unit-hypercube $-1 \leq [\boldsymbol{\xi}^{\mathcal{FF}}_{\text{normed},\alpha}]_i \leq 1$, for all components noted

$$\mathbf{S}\ \boldsymbol{\xi}^{\mathcal{FF}}_{\text{normed},\alpha} \leq \mathbf{1} \quad \Leftrightarrow \quad \mathbf{S}\ \boldsymbol{\Phi}\ \boldsymbol{\zeta}_\alpha \leq \mathbf{1} \quad \text{with} \quad \mathbf{S} = \begin{bmatrix} 1 & 0 & \cdots \\ -1 & 0 & \cdots \\ 0 & 1 & \cdots \\ 0 & -1 & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}_{2n_\theta \times n_\theta} \qquad \text{and} \quad \boldsymbol{\Phi}^{\text{-}1} = \boldsymbol{\Phi}^{\intercal}. \tag{23}$$

Accordingly, $2 \cdot n_\theta$ constraints have to be considered for the reduced basis in $[\mathbf{S}^{\text{red}}]_{2n_\theta \times n_\lambda}$. The component-wise design bounds of $\boldsymbol{\zeta}_\alpha$ can be computed, e.g. by the corresponding linear programming problem, by investigation of all diagonals of the MP – attained by evaluating Eq. (21) for all $2^{n_\theta - 1}$ corners of the centered unit-hypercube – or by reasonably estimating bounds. It is noted that the component-wise bounds of $\boldsymbol{\zeta}^{\text{red}}_\alpha$ in Eq. (23) are tighter, since MP diagonals are not all reachable in reduced expansions. However, since the eigenvectors are normed, the half diagonals of the centered unit-hypercube in orthogonal basis are general bounds $\left|[\boldsymbol{\zeta}^{\mathcal{FF}}_\alpha]_i\right| \leq \sqrt{n_\theta}$ per component.

In analogy to the decompositions in (Sudret and Der Kiureghian, 2000; Götz et al., 2016), the fraction $q = \sum_{i=1}^{n_\lambda} \lambda_i \,/\, \sum_{j=1}^{n_\theta} \lambda_j$ is an indicator for the ignorance of uncertainty in the approximated input space, sufficient for trade-off decisions between numerical effort and approximation quality. Finally, it is noted that, based on the affine interval transformations, the demonstrated mapping chain $\mathbb{X}^{\mathcal{I}} = \psi_{\text{i-X}}(\psi_{\zeta\text{-i,red}}(\zeta^{\text{red}}))$ is bijective. Hence, in contrast to the adverse sampling of the method investigated in Section 2.2, the novel MP-based fuzzy fields provide a uniqueness in sampling.

3. Numerical Example

Two numerical examples are presented to show the applicability of the novel method and for validation with documented results. The developed approach of Section 2.4 is implemented in Python and all numerical optimization tasks are performed with the optimization framework Pymoo (v0.6.1.6) of (Blank and Deb, 2020). For comparibility, the common evolutionary strategy is applied (single objective optimization), although the implementation is modified to enable brief hyper-parameter optimization via the Pymoo interface to the Optuna tools of (Akiba et al., 2019).

Example 1 – Validation by an MP-based Interval Field Problem
With (Jiang et al., 2015a, "Numerical Example 3") as reference, the polynomial interval function

$$\mathbb{G}^{\mathcal{I},\text{E1}}(\mathbf{X}^{\mathcal{I}}) = -10\,\mathbb{X}_1^{\mathcal{I}} + 4\,\mathbb{X}_2^{\mathcal{I}} - 12\,\mathbb{X}_3^{\mathcal{I}} + \mathbb{X}_1^{\mathcal{I}} \cdot \mathbb{X}_2^{\mathcal{I}} + 2\,\mathbb{X}_1^{\mathcal{I}} \cdot \mathbb{X}_3^{\mathcal{I}} + \mathbb{X}_2^{\mathcal{I}} \cdot \mathbb{X}_3^{\mathcal{I}} + 10 \tag{24}$$

is defined as basic solution for the uncertainty quantification. Additionally, the parameters are

$$\mathbf{I} = \begin{bmatrix} 1.0 & -0.4 & 0.4 \\ & 1.0 & -0.4 \\ \text{sym.} & & 1.0 \end{bmatrix}, \qquad \mathbb{X}_i^{\mathcal{I}} = [0,\, 8],\, i \in \{1,2,3\}, \tag{25}$$

where the matrix of interactions is according to Eq. (13) and all intervals have the same bounds.

The method definitions of the reference and the proposed method (see Section 2.4) differ in the transformation to the interacted intervals (cp. Eqs. (14) and (16)). However, Eq. (25) already has identical sums of row and column absolute values, and therefore, leads to a signed Sinkhorn form with Eq. (15). Hence, the methods are equivalent in the transformation.

The reference results are achieved by interval arithmetic with analytical interaction transformation of the intervals. The overestimation due to inherent interval dependency (wrapping effect) is mentioned in passing, but accepted without effort in (Jiang et al., 2015a). It is shown there that

$$\begin{aligned} \mathbb{G}^{\mathcal{I},\text{E1b}}_{\text{interacted}}(\boldsymbol{\xi}^{\mathcal{I}}) = 1.1851\,{\xi_1^{\mathcal{I}}}^2 - 2.3703\,{\xi_2^{\mathcal{I}}}^2 + 1.1851\,{\xi_3^{\mathcal{I}}}^2 + 2.9629\,\xi_1^{\mathcal{I}}\,\xi_2^{\mathcal{I}} + 5.9259\,\xi_1^{\mathcal{I}}\,\xi_3^{\mathcal{I}} \\ +2.9629\,\xi_2^{\mathcal{I}}\,\xi_3^{\mathcal{I}} - 0.8888\,\xi_1^{\mathcal{I}} + 7.1111\,\xi_2^{\mathcal{I}} - 1.7777\,\xi_3^{\mathcal{I}} - 30 \end{aligned} \tag{26}$$

results from the analytical interaction transformation. Due to interval wrapping, the equivalent is

$$\mathbb{G}^{\mathcal{I},\text{E1c}}_{\substack{\text{interacted}\\ \text{wrapping}}}(\boldsymbol{\xi}^{\mathcal{I}}_{\text{equiv.}}) = 1.1851\xi_1^{\mathcal{I}}\xi_2^{\mathcal{I}} - 2.3703\,\xi_3^{\mathcal{I}}\xi_4^{\mathcal{I}} + \cdots - 1.7777\,\xi_{15}^{\mathcal{I}} - 30. \tag{27}$$

Table I. Results of numerical example 1 with reference solutions of (Jiang et al., 2015a).

setup	*uncertainty quantification method*	*interaction method*	*wrapping avoided*	*basic solution*	*input bounds*	*resulting interval*	*relative radius*
E1a) reference	interval arithmetic	non-interacted	$\ominus$	Eq. (24)	$[0,\ 8]^3$	$[-198,\ 266]$	100.0 %
E1b) reference	interval arithmetic	Sec. 2.3, analytical	$\ominus$	Eq. (26)	$[-1,\ 1]^3$	$[-56.37,\ -3.63]$	11.4 %
E1c) validation	analytical bounds	Sec. 2.3, analytical	$\ominus$	Eq. (27)	$[-1,\ 1]^{15}$	$[-56.3698,\ -3.6302]$	11.4 %
E1d) computed	α-level optimized	Sec. 2.3 $\widehat{=}$ 2.4, full	✓	Eq. (24)	$[0,\ 8]^3$	$[-45.7037,\ -11.9259]$	7.3 %
E1e) computed	α-level optimized	Sec. 2.4, reduced ev.	✓	Eq. (24)	$[-1,\ 1]^2$	$[-37.74,\ -22.05]$	3.4 %

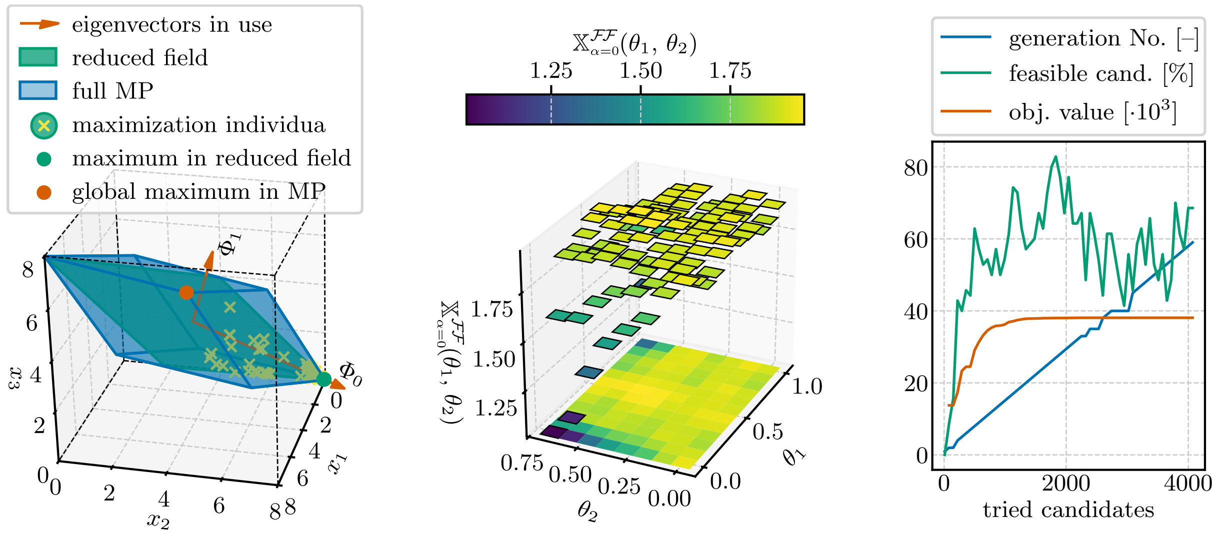
The numerical experiment setups and computed results are listed in Table I. The results of (Jiang et al., 2015a) are taken as references, denoted as setups E1a) and b). In validation setup E1c), the interval wrapping is tracked by interval optimization with the expanded form of the problem. The reference is successfully reproduced, although it is noted that the optima of Eq. (27) are analytically found at such bounds of the fifteen-dimensional input space, where variable combinations trivially satisfy positive/negative signs of all equation parts for maximization/minimization, respectively.

The proposed interaction method of MP-based fuzzy fields is demonstrated in E1d) and e). Therefore, the fuzzy field is viewed as an interval field (single α-level investigation) with predefined interaction matrix. In this specific case, as stated before, the proposed method and the reference method are equivalent and fully comparable. However, E1d) shows that the wrapping effect is avoidable by interval optimization. The full consideration of all interval dependencies results in a tight interval, contrarily to the overestimation in the interval arithmetic approach of E1b).

Finally, setup E1e) is performed with a reduced two-dimensional basis, applying the proposed orthogonal expansion. For this, the maximization individua are visualized in Figure 3. The global maximum in the MP (green volume) is at a vertex (orange point) that is not included in the reduced field (blue plane), spanned by the two eigenvectors (orange arrows) that correspond to the two largest eigenvalues. By this, it is emphasized that the proposed reduced orthogonal expansion is numerically advantageous, due to the reduced basis. The full MP is covered by the two-dimensional sub-space, based on the eigenvalues' weighting of the most pronounced extent of the MP. However, if the global maximum is not included in the reduced sub-space, as it is the case in E1e) shown in Figure 3a), the uncertainty quantification is prone to underestimation of the resulting interval. In conclusion, the reduction with E1e) covers less than half of the result of E1d), while the number of effective basic solution evaluations is 108|96 for hyper-parameter optimization and 101|242 for the min.|max. to reach 10 % tolerance in the objective function over the last 5 optimization generations.

Example 2 – MP-based Fuzzy Field evaluated at the Rosenbrock benchmark function
The Rosenbrock function, according to (Blank and Deb, 2020), is investigated as basic solution by

$$\mathbb{G}^{\mathcal{F},\mathrm{E2}}_{\mathrm{Rosenbrock}}\big(\mathbf{X}^{\mathcal{FF}}\big) = \sum_{i=1}^{n_\theta-1}\Big(100\cdot\big(\mathbf{X}^{\mathcal{FF}}(\boldsymbol{\theta}_{i+1}) - \mathbf{X}^{\mathcal{FF}}(\boldsymbol{\theta}_i)^2\big)^2 + (1-\mathbf{X}^{\mathcal{FF}}(\boldsymbol{\theta}_i)^2\Big). \tag{28}$$



a) Maximization individua of E1e) on reduced orthogonal expansion (plane), enveloped by full MP.

b) Exemplary realization of MP-based 2-dim. spatially interacting fuzzy field at local maximum of E2a) in $\alpha = 0$.

c) Optimization characteristics for maximization task E2b) – feasibility due to constraints.

Figure 3. Results and characteristics of the optimization tasks for the numerical examples.

Table II. Results of numerical example 2, by 96-dim. Rosenbrock function (Blank and Deb, 2020) as basic solution.

setup	*uncertainty quantification method*	*MP-based fuzzy field*	*basic solution*	*input quantities*	*resulting fuzzy quantity*	*relative radius at $\alpha = 0$*
E2a)	α-level optimized	full	Eq. (28)	$\langle 0.5,\, 1.5,\, 2.0\rangle^{96}$	$\langle 55.7,\, 5367.5,\, 25268.1\rangle$	66.2 %
E2b)	α-level optimized	reduced ev.	Eq. (28)	$[-1,\, 1]^3$	$\langle 3\cdot 10^{-5},\, 5367.5,\, 38089.4\rangle$	100.0 %

Therein, $\mathbf{X}^{\mathcal{FF}}$ is simulated as stationary MP-based fuzzy field, as defined in Section 2.4. The field constituting marginal fuzzy quantity is $\mathbb{X}^{\mathcal{F}} = \langle 0.5,\, 1.5,\, 2.0\rangle$ and the observation points are defined in a two-dimensional equidistant 12×8 grid, as shown in Figure 3b). A spatial auto-interaction by squared exponential kernel, see (Schietzold et al., 2019), is defined with interaction length parameter $\theta_{\mathrm{d}} = 0.1$, allowing for a high scattering in the field realizations. The results of both setups for this numerical example are listed in Table II. For the fuzzy analyses, α-level optimization for support $\alpha = 0$ is performed in addition to a single computation of the core $\alpha = 1$. The full MP-based fuzzy field with 96 base variables, one for each observation point, is simulated in E2a). In contrast, a reduced orthogonal basis is applied for E2b), by 3 eigenvectors ($q_{\mathrm{E2b)}} = \sum_{i=1}^{3}\lambda_i \,/\, \sum_{j=1}^{96}\lambda_j = 9.6\,\%$).

Based on 5800|5800 evaluations in hyper-parameter optimizations and 5820|15130 evaluations in the optimizations of min|max, E2a) gets stuck in local optima, terminating with less than 10 % objective function change in the last 10 generations. With the same optimization setup, E2b) converges after only 5730|4163 hyper-parameter optimization evaluations and further 1331|2326 calls, outperforming E2a) by $\approx 50\,\%$ increased result radius at $\alpha = 0$. Hence, the reduced expansion enables finding the optima at all. In Figure 3c), optimization characteristics of E2b) are plotted, where the fraction of feasible candidates fluctuates over generations, emphasizing the importance of efficient constraint implementation for the design variables (orthogonal bases), see Eq. (23).

4. Conclusions and Outlook

Realistic system safety assessment with spatially and temporally dependent uncertainty under severe lack of knowledge requires epistemic uncertain field formulations. Mathematical models and uncertainty quantification for such fields are discussed in this contribution. Especially, an existing fuzzy field method, defined by orthogonal decomposition, is investigated. Although, improvement suggestions for this method are provided, disadvantages still remain. Hence, a novel fuzzy field method, based on the existing concept of multidimensional parallelepipeds (MP) for interval interactions, is developed. The disadvantages of the existing method are eliminated in the novel approach. The developments enable a consistent distance-based auto-interaction structure, with free choice of an interaction kernel, where corners of the hyperrectangle of interval input spaces are bounding and reachable in contrast to ellipsoidal convex interaction models. Moreover, the proposed MP-based fuzzy fields allow for the development of orthogonal decomposition and reduced basis expansion, developed for efficient fuzzy analysis. The orthogonal expansion method enables to select a desired approximation quality of a fuzzy field by number of α-level optimization design variables, while still covering the most significant extents of the MP in a low-dimensional representation.

The demonstrated methods can straightforward be applied for engineering tasks, e.g. in structural analysis with basic solutions by FE simulations. The dimensionality of the fuzzy field formulation can be defined for systems with arbitrary n_θ-dimensional spatially and temporally dependent uncertainty. As limitation of the proposed MP-based fuzzy field method, it is noted that projections of an MP in the (i, j)-coordinate planes are not parallelograms and, especially, dependent on the number of observation points, which is required to be recognized in the generalization of data-based uncertainty modeling for auto-interaction structures. In conclusion, it is therefore noted that there is potential for further research into the development of additional fuzzy field interaction models.

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