Efficient robust topology optimization of large-scale continuum structures under load uncertainty

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Abstract: With the increasing complexity of structural design problems, robust topology optimization (RTO) faces increasingly severe computational efficiency challenges. The huge cost of large-scale structural analysis, compounded by repeated analyses required for uncertainty quantification, substantially increases the overall computational burden of RTO. To address this issue, this study proposes an efficient robust topology optimization framework based on discrete variable (RDVTO). The framework first applies direct probability integration (DPIM) with an iterative sequence sampling strategy to quantify uncertainties, enabling accurate statistics with minimal structural analyses. Subsequently, improvements are introduced to the multigrid preconditioned conjugate gradient method based on extended multiscale finite elements (EMsFEM-MGPCG), further enhancing structural analysis efficiency. For load uncertainty issues, the improved method employs matrix representation of load vectors, enabling the calculation of displacements for all representative points in a single computation. Finally, sequential approximate integer programming with trust region (SAIP-TR) is adopted to solve RDVTO problems, producing clear topology configurations. For 3D examples involving millions of degrees of freedom, the proposed method still achieves efficient solutions on a single computer, highlighting its strong potential for engineering applications.

Keywords: Robust topology optimization; EMsFEM-MGPCG acceleration; Large-scale structures; Direct probability integration method; Sequential approximate integer programming with trust region

1. Introduction

In continuum structural design, topology optimization provides an effective means of determining material distribution without relying on a predefined structural configuration. Although this capability has made topology optimization widely used, many existing formulations are still developed under deterministic assumptions, where material parameters, external loads, and boundary-related conditions are regarded as fixed inputs (Liu et al. 2020; Yang et al. 2022). Such assumptions are often inconsistent with practical engineering situations. In real applications, structural performance is inevitably affected by material dispersion, manufacturing imperfections, and stochastic loading environments (Lüthen et al. 2021; Wang and Dai, 2023). A design obtained by deterministic optimization may therefore exhibit reduced stiffness, degraded robustness, or insufficient reliability once these uncertainties appear in service. To improve the stability of optimized layouts against input perturbations, uncertainty effects should be embedded into the design process rather than treated as post-processing factors (Zhu et al. 2021). Among various uncertainty sources, load randomness is of particular significance, since it directly changes the structural response and thus exerts a notable effect on the global structural behavior (Jung and Cho, 2004). Robust topology optimization (RTO) provides a suitable framework for this purpose by describing uncertain parameters through statistical quantities and including these quantities in the optimization model (Beyer and Sendhoff 2007; Li et al. 2023). Motivated by this consideration, this work focuses on RTO of continuum structures involving load uncertainty.

Despite its engineering significance, the application of RTO to large-scale structures remains computationally demanding. The difficulty is not caused by topology optimization alone, but by the coupling between large-scale finite element analysis and uncertainty quantification. On the one hand, each optimization iteration usually requires the assembly and solution of high-dimensional finite element equations, which becomes expensive as the design domain is refined (Baiges et al. 2019); (Herrero-Pérez et al. 2024). On the other hand, the evaluation of statistical responses under uncertainty often relies on repeated structural analyses, leading to a rapid increase in computational burden when accurate moment estimation is required (Martínez-Frutos and Herrero-Pérez, 2016). Therefore, an efficient RTO strategy should simultaneously reduce the number of structural evaluations required for uncertainty propagation and accelerate the solution of the finite element systems involved in each evaluation.

The multigrid preconditioned conjugate gradient method based on extended multiscale finite elements (EMsFEM-MGPCG) method has been demonstrated to be efficient for deterministic topology optimization (Liang, 2021); however, its direct extension to RTO does not fully remove the computational bottleneck. In the case of load uncertainty, numerous representative load samples may still have to be analyzed during each optimization iteration. As a result, repeated calls to the EMsFEM-MGPCG solver can dominate the total computational time, particularly for large-scale models. When the mesh size increases, this repeated reconstruction can become a major source of overhead and may even offset the acceleration gained from the multiscale solver.

To address these issues, this study develops an RTO framework that integrates direct probability integration method (DPIM) with an improved EMsFEM-MGPCG strategy. DPIM is adopted to evaluate the statistical moments of structural responses efficiently using a limited number of representative analyses (Chen and Yang, 2019). Based on the type of uncertainty considered, the EMsFEM-MGPCG procedure is further modified to reduce unnecessary repeated computations. For load uncertainty, the responses corresponding to multiple load cases are solved in a unified matrix form, thereby avoiding repeated solver calls within a single optimization iteration. The resulting robust discrete-variable topology optimization problem is solved using sequential approximate integer programming with trust region (SAIP-TR), enabling the direct generation of clear black-and-white structural layouts.

2. RDVTO Formulation

Typically, the objective function of RDVTO is a weighted sum of the mean and standard deviation for the structural response of interest, and the structural volume serves as the constraint. In this study, the formulation of robust topology optimization is defined as

$$\begin{aligned} \min_{\rho} \quad & C_R(\rho) = \mathrm{E}[C(\rho)] + w\mathrm{Std}[C(\rho)] \\ \text{s.t.:} \quad & \begin{cases} \mathbf{K}(\boldsymbol{\rho},\boldsymbol{\Theta})\mathbf{U} = \mathbf{F}(\boldsymbol{\Theta}) \\ \sum_{i=1}^{N} \rho_i V_i \le \overline{V_f} V_0 \\ \rho_i \in \{0,1\} \quad , \quad i = 1,2,\ldots N \end{cases} \end{aligned} \tag{1}$$

where the compliance C is expressed as $C(\rho)=\mathbf{F}^T\mathbf{U}$. E[$C(\rho)$] and Std[$C(\rho)$] are the mean and standard deviation of the compliance C, respectively. In this study, w is set to 3. $\boldsymbol{\rho}$ denotes the N-dimensional elemental density vector, and ρ_i is the density of the ith element, which takes the value of 0 or 1. $\boldsymbol{\Theta} = [\Theta_1, \Theta_2, ..., \Theta_n]$ is the n-dimensional random variable vector. **K** means the global stiffness matrix, **F** and **U** denote the external load vector and the global displacement vector. V_i indicates the volume of the ith element, V_0 is the volume of the structure when full of material, and $\overline{V_f}$ is the volume fraction. In this article, we use the discrete variable topology optimization method SAIP-TR to solve the optimization equation

In the SAIP-TR method, the objective and constraint functions of the programming subproblem are constructed using the first-order Taylor expansion. Specifically, a kth sequential integer programming subproblem of the SAIP-TR method is formulated by

$$\min_{\rho} \quad C_R(\rho) \approx C_R(\rho^k) + \sum_{i=1}^{N} \frac{\partial C(\rho^k)}{\partial \rho}(\rho - \rho^k)$$

$$\text{s.t.:} \begin{cases} \mathbf{K}(\boldsymbol{\rho}^k, \boldsymbol{\Theta})\mathbf{U} = \mathbf{F}(\boldsymbol{\Theta}) \\ \sum_{i=1}^{N} \rho_i^k V_i \le \overline{V_f} V_0 \\ \sum_{i=1}^{N} r_i \left(\rho_i - \rho_i^k\right) \le r_k \quad , \quad r_i = \{-1, 1\} \\ \rho_i = \{0, 1\} \quad , \quad i = 1, 2, \ldots N \end{cases} \tag{2}$$

where the penultimate constraint denotes the trust region, and the parameter r_k represents the radius of the trust region for the kth subproblem. Physically, it indicates the number of elements allowed to change density in each iteration, $\boldsymbol{\rho}^k$ represents the design variable vector of the kth iteration. Moreover, the above integer programming subproblems are solved by the canonical relaxation algorithm.

3. Uncertainty quantification for RDVTO

Calculation of statistical moments is crucial for RTO. However, there is no unified method for solving the high-dimensional random variables RTO problems efficiently and accurately. Therefore, the DPIM (Chen and Yang, 2019), a novel and efficient stochastic analysis method for stochastic static and dynamic systems, is employed to calculate the statistical moments of RTO problems herein.

In the RDVTO problems, the input random variables and the structural compliance response can be defined as the following mapping

$$Q: Y = C(\boldsymbol{\rho}, \boldsymbol{\Theta}) \tag{3}$$

According to the principle of probability conservation (Syski, 1967; Soong and Bogdanoff, 1973; Li and Chen, 2009), the probability measure determined by the random sources remains invariant in the propagation process of randomness from system input to output, i.e.

$$\int_{\Omega_Y} p_Y(\mathbf{y}, t)\,\mathrm{d}\mathbf{y} = \int_{\Omega_{\boldsymbol{\Theta}}} p_{\boldsymbol{\Theta}}(\boldsymbol{\theta})\,\mathrm{d}\boldsymbol{\theta} \tag{4}$$

where $p_Y(\boldsymbol{y})$ and $p_{\boldsymbol{\Theta}}(\boldsymbol{\theta})$ denote the joint probability density function (PDF) of Y and $\boldsymbol{\Theta}$, respectively. Ω_Y and $\Omega_{\boldsymbol{\Theta}}$ denote the sample spaces of the input vector and output vector.

Based on the sifting properties of the Dirac δ function (Li and Chen, 2008) and Eq.(4), the relationship between the PDF of output response Y=C($\boldsymbol{\rho}$, $\boldsymbol{\theta}$) and the input PDF (Li et al. 2021; Peng et al. 2024) is derived as

$$p_Y(y) = \int_{\Omega_{\boldsymbol{\Theta}}} \delta\left(y - C(\boldsymbol{\rho}, \boldsymbol{\theta})\right) p_{\boldsymbol{\Theta}}(\boldsymbol{\theta}) \mathrm{d}\boldsymbol{\theta} \tag{5}$$

Eq. (3) is named as probability density integral equation of static system (Li and Chen, 2008). $\delta\left(y - C(\boldsymbol{\rho}, \boldsymbol{\theta})\right)$ is a Dirac δ function, which is a linear functional and denotes the conditional PDF of stochastic response y given random input vector $\boldsymbol{\theta}$, i.e. $p_{Y|\boldsymbol{\Theta}}(y \mid \boldsymbol{\theta}) = \delta(y - C(\boldsymbol{\rho}, \boldsymbol{\theta}))$. Eq. (5) is a total probability formula of density form, namely

$$p_Y(y) = \int_{\Omega_{\boldsymbol{\Theta}}} p_{Y|\boldsymbol{\Theta}}(y \mid \boldsymbol{\theta}) p_{\boldsymbol{\Theta}}(\boldsymbol{\theta}) \mathrm{d}\boldsymbol{\theta} = \int_{\Omega_{\boldsymbol{\Theta}}} \delta\left(y - C(\boldsymbol{\rho}, \boldsymbol{\theta})\right) p_{\boldsymbol{\Theta}}(\boldsymbol{\theta}) \mathrm{d}\boldsymbol{\theta} \tag{6}$$

Meanwhile, Eq. (5) implies that the response PDF is the expectation of conditional PDF of response given random input $\boldsymbol{\theta}$ and the expectation of Dirac δ function, namely

$$p_Y(y) = \mathrm{E}\left[p_{Y|\boldsymbol{\Theta}}(y \mid \boldsymbol{\theta})\right] = \mathrm{E}\left[\delta\left(y - C(\boldsymbol{\rho}, \boldsymbol{\theta})\right)\right] \tag{7}$$

The ith moment $m_i[Y]$ of the stochastic response Y is defined by

$$m_i[Y] = \mathrm{E}(Y^i) = \int_{\Omega_Y} y^i p_Y(y) \mathrm{d}y \tag{8}$$

In general, the response PDF $p_Y(y)$ in Eq. (11) is difficult to calculate. Thus, substituting in Eq. (5) into Eq. (8) yields

$$m_i\left[Y\right] = \int_{\Omega_Y} \int_{\Omega_{\boldsymbol{\Theta}}} y^i \delta\left(y - C(\boldsymbol{\rho}, \boldsymbol{\theta})\right) p_{\boldsymbol{\Theta}}(\boldsymbol{\theta}) \mathrm{d}\boldsymbol{\theta} \mathrm{d}y = \int_{\Omega_{\Theta}} \left\{ \int_{\Omega_Y} y^i \delta\left(y - C(\boldsymbol{\rho}, \boldsymbol{\theta})\right) dy \right\} p_{\boldsymbol{\Theta}}(\boldsymbol{\theta}) \mathrm{d}\boldsymbol{\theta} \tag{9}$$

Using the sifting property of the Dirac function, the ith order origin moment of the stochastic response Y in input probability space is expressed as (Yang et al. 2025)

$$m_i\left[Y\right] = \int_{\Omega_{\boldsymbol{\Theta}}} \left(C(\boldsymbol{\rho}, \boldsymbol{\theta})\right)^i p_{\boldsymbol{\Theta}}(\boldsymbol{\theta}) \mathrm{d}\boldsymbol{\theta} \tag{10}$$

Eq. (10) usually involves high-dimensional integrals and is hard to obtain an exact solution. Based on the representative point selection strategy of GF-discrepancy, DPIM is utilized to compute Eq. (10) by probability space partition, which is discretized as

$$m_i\left[Y\right] = \int_{\Omega_{\boldsymbol{\Theta}}} \left(C(\boldsymbol{\rho}, \boldsymbol{\theta})\right)^i p_{\boldsymbol{\Theta}}(\boldsymbol{\theta}) \mathrm{d}\boldsymbol{\theta} = \sum_{q=1}^{M} \left(C(\boldsymbol{\rho}, \boldsymbol{\theta}_q)\right)^i P_q \tag{11}$$

where M is the number of sample points required for structural analysis, and P_q denotes the assigned probability of the qth representative point $\boldsymbol{\theta}_q$ in the corresponding representative region (Chen et al. 2016). The number of sample points relates to the computation cost of structural analysis. In this paper, an iterative sampling strategy (Tao et al. 2022) is employed to efficiently determine the number of sample points.

In the iterative sampling strategy, the GF-discrepancy based point selection approach (Chen et al. 2016) is firstly used to obtain a pool of candidate sample points. Then, the sample points are gradually selected from the pool until the proposed stopping criterion is satisfied. In this study, the stopping criteria are defined below

$$N_u = tol_{N_u} \tag{12}$$

where

$$N_\mu = \sum_{i=1}^{n} I\left(\varepsilon_{\mu i} \le tol_{\varepsilon_\mu}\right);\ \varepsilon_\mu = \frac{\left\|\mu_i - \mu_{i-1}\right\|_2}{\left\|\mu_{i-1}\right\|_2} \tag{13}$$

where $I(x)$ is indicator function, $I = 1$ when x satisfies the requirement, otherwise $I = 0$. N_μ represent the total times when the error $\varepsilon_{\mu i}$ is less than the threshold tol_{ε_μ} in the first n iterations. In this study, the threshold is set to 1×10^{-3}. μ_i indicates the weighted sum of the mean and standard deviation of stochastic response. tol_{Nu} is a positive integer and the sampling process stops when Eq. (12) is satisfied. This strategy in DPIM can adaptively determine the number of samplings, which significantly enhances the efficiency. Interested readers can refer to Ref. (Tao et al. 2022) for details. According to Eq. (11), the first and second order origin moments are obtained by using DPIM

$$\mathrm{E}(Y) = \int_{\Omega_{\boldsymbol{\Theta}}} C(\boldsymbol{\rho},\boldsymbol{\theta}) p_{\boldsymbol{\Theta}}(\boldsymbol{\theta})\mathrm{d}\boldsymbol{\theta} = \sum_{q=1}^{M} C(\boldsymbol{\rho},\boldsymbol{\theta}_q) P_q \tag{14}$$

$$\mathrm{E}[Y^2] = \int_{\Omega_{\boldsymbol{\Theta}}} \left(C(\boldsymbol{\rho},\boldsymbol{\theta})\right)^2 p_{\boldsymbol{\Theta}}(\boldsymbol{\theta})\mathrm{d}\boldsymbol{\theta} = \sum_{q=1}^{M} \left(C(\boldsymbol{\rho},\boldsymbol{\theta}_q)\right)^2 P_q \tag{15}$$

The standard deviation of stochastic response Y can then be obtained by

$$\begin{aligned}\mathrm{Std}(Y) &= \sqrt{\int_{\Omega_{\boldsymbol{\Theta}}} \left(C(\boldsymbol{\rho},\boldsymbol{\theta})\right)^2 P_q \mathrm{d}\boldsymbol{\theta} - \left(\int_{\Omega_{\boldsymbol{\Theta}}} C(\boldsymbol{\rho},\boldsymbol{\theta}) P_q \mathrm{d}\boldsymbol{\theta}\right)^2} \\ &= \sqrt{\sum_{q=1}^{M} \left(C(\boldsymbol{\rho},\boldsymbol{\theta}_q)\right)^2 P_q - \left(\sum_{q=1}^{M} C(\boldsymbol{\rho},\boldsymbol{\theta}_q) P_q\right)^2}\end{aligned} \tag{16}$$

4. Improved EMsFEM-MGPCG

EMsFEM-MGPCG was originally incorporated into the SAIP-TR topology optimization scheme by Liang (2021), where it enabled high-resolution stiffness-oriented designs with low-order finite elements and small filtering radii. However, that work was mainly developed for deterministic optimization. Once uncertainties in external loads are introduced, the original implementation no longer fully reflects the potential efficiency of EMsFEM-MGPCG. Therefore, this section reformulates the method for RDVTO problems involving load uncertainty, with the goal of lowering the repeated analysis cost.

The main computational burden in RDVTO comes from the large number of finite element analyses required to evaluate statistical responses and their sensitivities. For load uncertainty, a conventional procedure performs an independent equilibrium analysis for each representative load at every design iteration, although these analyses use the same stiffness matrix within the current iteration. If EMsFEM-MGPCG is used in this manner, the solver still has to be called repeatedly for different representative points. To avoid this inefficiency, the proposed modification rewrites the problem in a multi-right-hand-side form. The load vectors associated with all representative points are grouped together, and the corresponding displacement solutions are obtained collectively under the shared stiffness matrix. By reusing the stiffness information across all load cases, the improved EMsFEM-MGPCG eliminates unnecessary repeated solves and substantially accelerates RDVTO under load uncertainty.

For M representative uncertainty points, the conventional independent finite element analysis is formulated as:

$$Ku_m = f_m \qquad m = 1, 2, \ldots, M \tag{17}$$

Although these independent analyses share the exact same structural stiffness matrix K within the current iteration, a standard sequential solver executes M distinct linear solution pipelines. To resolve this inefficiency, the governing equations are rewritten into a compact multi-right-hand-side block matrix form:

$$KU = F \qquad U = [u_1, u_2, \ldots, u_M], \;\; F = [f_1, f_2, \ldots, f_M] \tag{18}$$

In the EMsFEM paradigm, the global fine-mesh stiffness matrix is projected onto a downscaled coarse-mesh subspace via numerical basis functions: $K_c = \mathrm{P}^T K \mathrm{P}$. By utilizing the Eq. (18) formulation, the costly base-function calculation, fine-to-coarse projection, and multigrid preconditioning smoothers are executed exactly once per SAIP-TR optimization step. The block system is then solved collectively using parallelized MGPCG Krylov subspace projections, eliminating all redundant sub-meshing and assembly loops.

The solution framework is shown in Fig. 1.

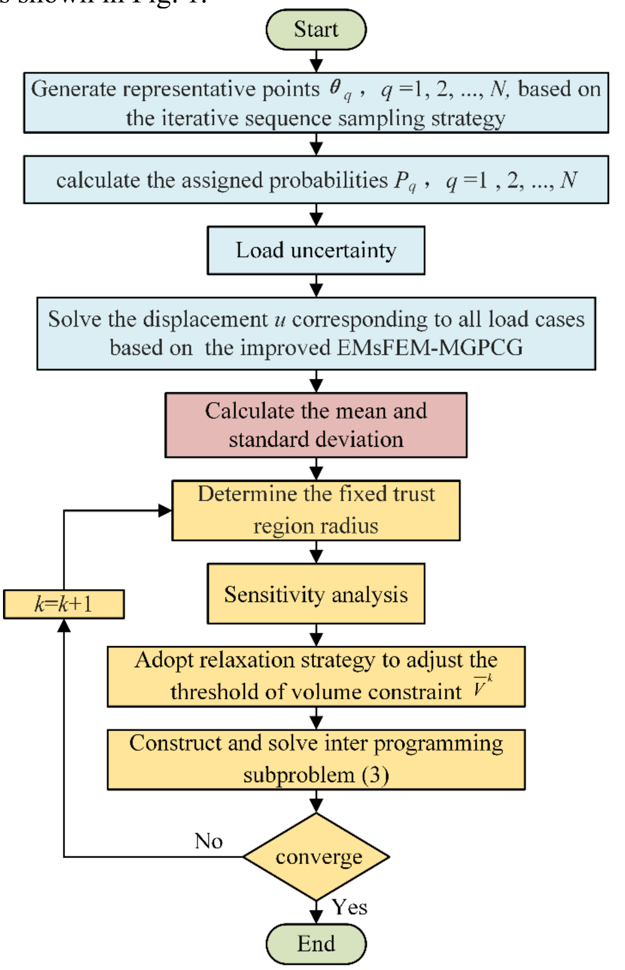

Fig. 1 Schematic diagram based on the proposed framework

5. Numerical examples

Engineering bridge structures are inherently three-dimensional, and their numerical models usually involve far more degrees of freedom than two-dimensional benchmarks. As the mesh resolution increases, the repeated structural analyses required in RTO become increasingly expensive. To examine the applicability of the proposed method to practical large-scale problems, a three-dimensional bridge model is therefore adopted as the test case.

The bridge deck is subjected to uncertain spatial loading, which may arise from variations in vehicle positions, traffic paths, snow or ice accumulation, and wind effects. The corresponding boundary conditions are shown in Fig. 2. The design domain is discretized by a 30 × 15 × 6 coarse mesh, and each coarse element is further resolved into 6 × 6 × 6 fine elements. The prescribed volume fraction is $V_f = 0.1$. A distributed load with random magnitude and direction is imposed on the non-design region marked in Fig. 2. The load magnitude is represented by a Gaussian random field constructed using EOLE, with a mean value of 1, a standard deviation of 0.1, and a correlation length of 30. Five truncation terms are retained, keeping the truncation error below 2%. The load direction follows a prescribed normal distribution, and 80 representative points selected by the iterative sequence sampling strategy are adopted.

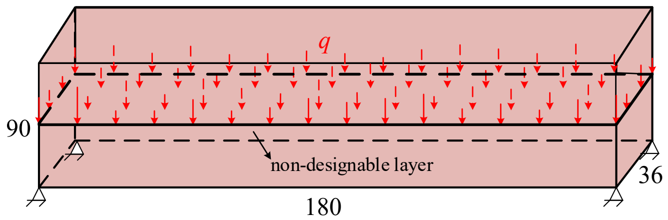

Fig. 2 3D bridge structure under random distributed loads with uncertain magnitude and direction

The three-dimensional bridge example involves 1,828,281 degrees of freedom, making it a typical million-scale RDVTO problem. The optimized layout in Fig. 3 captures the main load-carrying pattern of the real bridge shown in Fig. 4, while retaining fine structural details. Since SAIP-TR treats the design variables in a discrete manner, the final topology is directly obtained as a well-defined black-and-white configuration, without the need for filtering or post-processing. To assess the computational advantage of the proposed framework, the direct solver is used as a reference. Its cost is prohibitive: a single structural analysis iteration requires nearly 20 hours, so a complete optimization process would be impractical for engineering use. In comparison, the proposed framework reduces the structural analysis time per iteration to 82 s. The whole optimization process reaches convergence after 110 iterations and takes approximately 2.5 hours on a single workstation. These results suggest that the proposed method enables efficient and accurate topology optimization for million-DOF three-dimensional structures, and offers a practical computational strategy for large-scale engineering design.

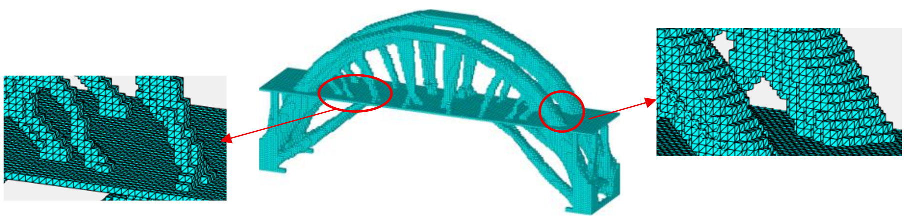

Fig. 3 Optimal topology configuration of the 3D bridge by RDVTO

Fig. 4 An existing through-arch bridge

6. Conclusions

This study develops an efficient RDVTO framework by improved EMsFEM-MGPCG and coupling it with DPIM and SAIP-TR. the modified solver reduces the cost of structural analysis under load uncertainty, while DPIM limits the number of analyses required for statistical moment estimation. For load uncertainty, the improved EMsFEM-MGPCG method enables simultaneous solution of all representative point displacements using a single right-hand-side matrix and one factorization of the global stiffness matrix K. For 3D RDVTO problems involving millions of degrees of freedom, the proposed framework reduces the structural analysis time per iteration from approximately 20 hours using traditional methods to just 82 s. This efficiency gain is primarily attributed to the improved EMSFEM-MGPCG's capability to solve multiple load cases simultaneously in matrix form, thereby eliminating redundant solver calls. Consequently, the framework achieves remarkable computational performance and demonstrates strong engineering applicability for large-scale structural optimization.

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