A Bivariate Distribution Based on Squared Normal Transformation

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Abstract. Modeling non-Gaussian dependent random variables remains challenging in structural reliability analysis. This study proposes a bivariate squared normal (BSN) distribution based on the squared normal transformation (SNT). The joint cumulative distribution function (JCDF) is constructed by decomposing the probability into marginal distributions and a dependence function, both modeled using SNT. A power transformation with an optimized parameter is introduced to align the kurtosis of the transformed variable with that of a squared normal reference distribution, improving accuracy for strongly non-Gaussian cases. The joint probability density function (JPDF) is obtained by applying a central difference scheme to the JCDF. Validation against typical bivariate distributions, including classical and copula-based models, shows that the BSN distribution achieves high JCDF accuracy and acceptable JPDF accuracy, demonstrating strong flexibility and robustness across diverse dependence structures.

Keywords: Bivariate probability distribution; Squared normal transformation; Bivariate squared normal distribution; Non-Gaussian marginal, non-Gaussian dependency

1. Introduction

Probability distributions are essential in civil engineering for structural reliability analysis, risk assessment, and performance-based design (Li et al., 2019). Many engineering problems involve dependent non-Gaussian random variables, making bivariate probability distributions (BPDs) a necessary foundation for higher-dimensional modeling (Sklar, 1973).

Existing BPD construction methods fall into two categories: parameter-based and dependence-function-based (Tang et al., 2013). Parameter-based distributions (e.g., bivariate normal, lognormal, Gumbel (Gumbel, 1960), Weibull (Weibull, 1951)) are mathematically tractable but often require identical marginal types, limiting their flexibility (Zentner, 2017). Dependence-function-based models, particularly copulas, allow arbitrary marginals but require time-consuming selection of both marginals and dependence functions (Tang et al., 2015; Wang and Li, 2018; Goda, 2010; Durante and Jaworski, 2010).

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Moment-based transformation methods construct probability distributions directly from statistical moments. Li et al. (Li et al., 2025) proposed a bivariate cubic normal (BCN) distribution, where both marginal and dependence structures are modeled using a cubic normal transformation based on the first four moments and mixed moments. While flexible, this approach suffers from high computational cost.

To address these issues, this paper adopts the squared normal transformation (SNT) proposed by Zhao and Ono (Zhao and Ono, 2000) and further simplified by Qin et al. (Qin et al., 2020). The SNT requires only the first three marginal moments (mean, standard deviation, skewness) while still incorporating mixed moments for dependence modeling. This reduces moment requirements, simplifies computation, improves numerical stability with smaller samples, and preserves variable interpretability. Based on this, we propose a bivariate squared normal (BSN) distribution. The joint CDF is constructed using probability decomposition, with marginals and dependence both modeled by SNT. A power transformation with an optimized parameter handles strong non-Gaussian dependence by aligning the transformed variable with the SNT distributional form. The joint PDF is obtained using central difference of the JCDF.

The main contributions are: (1) a bivariate distribution model based on first three marginal moments with mixed-moment dependence modeling; (2) a transformation-based strategy to improve non-Gaussian robustness; (3) numerical implementation of the JPDF using central difference; and (4) validation on representative bivariate distributions.

2. Construction of bivariate squared normal distribution

Without loss of generality, a two-dimensional vector $[X, Y]$ is investigated, where X and Y are two random variables existing within a sample space S. For any x and $y \in \mathbb{R}$, where $\mathbb{R}$ represents the real numbers field, the JCDF of the BPD is defined as the following probability:

$$F(x, y) = \Pr[(X \leq x) \cap (Y \leq y)], \tag{1}$$

Based on probability theory, Eq. 1 can be expanded as (Li et al., 2025):

$$F(x, y) = \frac{1}{2}\left[F_1(x) + F_2(y) - \delta(x, y)\right], \tag{2}$$

where $F_1(\cdot)$ and $F_2(\cdot)$ are the marginal CDFs of X and Y, and $\delta(x, y)$ is the dependence function defined as:

$$\delta(x, y) = \Pr[(X - x)(Y - y) \leq 0]. \tag{3}$$

Eq. 2 is a probability identity that holds for any bivariate distribution. Based on this decomposition, the construction of JCDF includes two primary objectives: (1) modeling the marginal distributions $F_1(x)$ and $F_2(y)$; and (2) evaluating the dependence function $\delta(x, y)$. These two objectives are discussed in the following subsections.

2.1. Model of the marginal distributions

When marginal distributions are unknown, the squared normal transformation (SNT) based on the first three moments is employed (Zhao and Ono, 2000; Qin et al., 2020). For a random variable ψ with moments $\Theta = \{\mu, \sigma, \alpha_3\}$, its CDF is:

$$F(\psi) = \Phi\left[u(\psi, \Theta)\right], \tag{4}$$

where $\Phi(\cdot)$ is the cumulative distribution function (CDF) of a standard normal random variable; $\Theta_i = \{\mu_i, \sigma_i, \alpha_{3i}\}$ represents the first three moments of the i-th random variable, with $i = 1$ and 2 representing X and Y, respectively: μ_i is the mean, σ_i is the standard deviation, α_{3i} is the skewness; and $u(\cdot, \cdot)$ is the squared normal transformation function, which maps the standardized random variable to a standard normal variable through a quadratic polynomial (Qin et al., 2020).

The standardized variable $x_s = (\psi - \mu_i)/\sigma_i$ is expressed as a quadratic function of a standard normal variable u:

$$x_s = a_{1i} + a_{2i}u + a_{3i}u^2, \tag{5}$$

Following the simplified third-moment normal transformation (Qin et al., 2020), the coefficients are expressed as explicit functions of the skewness α_{3i}:

$$a_{3i} = \frac{\sqrt{3}}{10}\alpha_{3i}, \quad a_{2i} = 1 - \frac{\sqrt{3}}{50}\alpha_{3i}^2, \quad a_{1i} = -a_{3i}. \tag{6}$$

The inverse transformation (from x_s to u) is obtained by solving the quadratic equation:

$$u = \frac{-a_{2i} + \sqrt{a_{2i}^2 - 4a_{3i}(a_{1i} - x_s)}}{2a_{3i}}, \quad \alpha_{3i} \neq 0. \tag{7}$$

When the skewness $\alpha_{3i} = 0$, Eq. 7 simplifies to $u = x_s$, which is consistent with the standard normal transformation. A key advantage of SNT is that it requires only the first three moments, avoiding the need for kurtosis, while providing an explicit polynomial form that is computationally efficient and numerically stable. The applicable range of the simplified SNT is $|\alpha_{3i}| \leq 2$, which covers most engineering applications.

2.2. Evaluation of the dependence function

Treating $G(X, Y) = (X - x)(Y - y)$ as a random variable Z, the dependence function is $\delta(x, y) = F_Z(0)$. Using the method of moments (Zhao and Lu, 2021), combined with the SNT framework, Z is approximated by the squared normal transformation, yielding:

$$\delta(x, y) = \Phi\left[u(0, \Theta_Z)\right], \tag{8}$$

where $\Theta_Z = \{\mu_Z, \sigma_Z, \alpha_{3Z}\}$ are the first three moments of Z, computed as:

$$\mu_Z = E[XY] - xE[Y] - yE[X] + xy, \tag{9}$$

$$\sigma_Z = \sqrt{E[(Z-\mu_Z)^2]}, \tag{10}$$

$$\alpha_{3Z} = \frac{E[(Z-\mu_Z)^3]}{\sigma_Z^3}. \tag{11}$$

These expectations involve mixture moments of X and Y.

Typically, $G(X,Y)$ exhibits strong non-Gaussian behavior and the accuracy of existing distribution models is insufficient. To obtain an appropriate estimation for the strong non-Gaussian case, a power transformation is introduced. Let

$$\tilde{G}(X,Y) = \operatorname{sgn}(G) \cdot |G|^q, \quad (q > 0), \tag{12}$$

where $\operatorname{sgn}(\cdot)$ denotes the sign function. This transformation preserves the sign of G while adjusting its kurtosis to match that of a squared normal reference distribution when q is appropriately chosen.

To determine the optimal q, we first construct a reference distribution for Z using the squared normal transformation based solely on its first three moments Θ_Z. From this reference distribution, the target kurtosis κ_{target} is computed using Gauss-Hermite quadrature (Abramowitz and Stegun, 1964). Specifically, let u_i and w_i be the nodes and weights of the Gauss-Hermite quadrature, and obtain the corresponding samples of the standardized variable using the inverse SNT:

$$z_i = u_{SNT}^{-1}(u_i; \Theta_Z), \tag{13}$$

where $u_{SNT}^{-1}(\cdot; \Theta_Z)$ denotes the inverse squared normal transformation defined in Eq. 7. The target kurtosis is then computed as:

$$\kappa_{target} = \frac{\sum_i w_i (z_i - \mu_Z)^4}{\left(\sum_i w_i (z_i - \mu_Z)^2\right)^2}. \tag{14}$$

This target kurtosis represents the fourth-moment characteristic of the distribution that is consistent with the first three moments Θ_Z under the SNT framework.

The optimal power parameter q is then obtained by minimizing the squared difference between the kurtosis of the transformed variable $\tilde{Z} = \tilde{G}(X,Y)$ and this target kurtosis:

$$\min_{q>0} \left(\kappa_{\tilde{Z}}(q) - \kappa_{target}\right)^2, \tag{15}$$

where $\kappa_{\tilde{Z}}(q)$ is the kurtosis of $\tilde{Z}$. This approach, analogous to the Box-Cox transformation in entropy-based fitting, ensures that the transformed variable $\tilde{Z}$ aligns with the reference distribution in terms of kurtosis, while the dependence function $\delta(x,y)$ is subsequently evaluated using the SNT based on the first three moments of $\tilde{Z}$:

$$\delta(x,y) = \Phi\left[u(0, \Theta_{\tilde{Z}})\right]. \tag{16}$$

The optimization in Eq. 15 is a one-dimensional bound-constrained problem ($q > 0$), which can be efficiently solved using standard algorithms such as the simplex search method (Mehta and Chowdhury, 2012). Once the optimal q is determined, the moments $\Theta_{\tilde{Z}} = \{\mu_{\tilde{Z}}, \sigma_{\tilde{Z}}, \alpha_{3\tilde{Z}}\}$ are computed from $\tilde{Z}$, and the dependence function is evaluated using Eq. 16.

2.3. JPDF based on central difference

The joint PDF of the BSN distribution is derived from the JCDF by second-order mixed differentiation:

$$f(x,y) = \frac{\partial^2 F(x,y)}{\partial x \partial y}. \tag{17}$$

Thus, once the JCDF $F(x,y)$ is constructed, the corresponding JPDF can be obtained directly from its mixed partial derivative. Since $F(x,y)$ is evaluated numerically rather than expressed analytically, the mixed partial derivative is approximated using a second-order central difference scheme:

$$f(x_i, y_j) \approx \frac{F_{i+1,j+1} - F_{i+1,j-1} - F_{i-1,j+1} + F_{i-1,j-1}}{4\Delta x \Delta y}, \tag{18}$$

where $F_{i\pm1,j\pm1} = F(x_i \pm \Delta x, y_j \pm \Delta y)$. The step sizes are set as $\Delta x = \epsilon \cdot \sigma_X$, $\Delta y = \epsilon \cdot \sigma_Y$ with $\epsilon = 10^{-3}$ to 10^{-4}.

Since both marginals and dependence are modeled using SNT based only on the first three moments, the proposed model is named the bivariate squared normal (BSN) distribution.

3. Flexibility in data fitting

A flexible distribution should have the capability to approximate commonly used distributions. In this section, the proposed BSN distribution is applied to simulate several commonly used bivariate probability distributions. To balance comprehensiveness and conciseness, three representative distributions are selected: one classical distribution (bivariate normal), one strongly non-Gaussian distribution (bivariate Weibull), and one dependence function-based distribution (BPD with Clayton copula and lognormal marginals).

Four cases are considered as follows:

1. Case 1: $\rho = 0$ and $\{\mu_1, \mu_2, \sigma_1, \sigma_2\} = \{2, 2, 0.05, 0.05\}$
2. Case 2: $\rho = 0$ and $\{\mu_1, \mu_2, \sigma_1, \sigma_2\} = \{2, 2, 0.1, 0.1\}$
3. Case 3: $\rho = 0.5$ and $\{\mu_1, \mu_2, \sigma_1, \sigma_2\} = \{2, 2, 0.05, 0.05\}$
4. Case 4: $\rho = 0.5$ and $\{\mu_1, \mu_2, \sigma_1, \sigma_2\} = \{2, 2, 0.1, 0.1\}$

The parameters of the marginal distributions for different cases are listed in Table I, along with the corresponding mean values and standard deviations.

Table I. Parameters of the marginal distributions of X and Y.

Distribution	X				Y			
	α_x	β_x	μ_1	σ_1	α_y	β_y	μ_2	σ_2
Normal	2	0.05	2	0.05	2	0.05	2	0.05
	2	0.10	2	0.10	2	0.10	2	0.10
Weibull	49.3883	2.0228	2	0.05	49.3883	2.0228	2	0.05
	24.8749	2.0443	2	0.10	24.8749	2.0443	2	0.10
Clayton-Lognormal	0.6928	0.0250	2	0.05	0.6928	0.0250	2	0.05
	0.6919	0.0500	2	0.10	0.6919	0.0500	2	0.10

Table II. Moments of Z and $\tilde{Z}$ with optimal q when $x = 2$ and $y = 2$

Distribution	Case	μ_Z	σ_Z	α_{3Z}	κ_{target}	q	$\mu_{\tilde{Z}}$	$\sigma_{\tilde{Z}}$	$\alpha_{3\tilde{Z}}$	$\alpha_{4\tilde{Z}}$
Normal	1	0	0.0025	0.0004	3.0000	0.5788	0	0.0252	-0.0015	3.0000
	2	0	0.0050	-0.0162	3.0004	0.5791	0	0.0379	-0.0023	3.0004
	3	0.0012	0.0028	2.3207	6.4684	0.7934	0.0186	0.0378	0.1589	6.4684
	4	0.0025	0.0056	2.3350	6.4699	0.7916	0.0263	0.0534	1.3087	6.4699
Weibull	1	0	0.0026	0.8431	4.0210	0.6107	0.0002	0.0199	0.0582	4.0210
	2	0	0.0050	0.6231	3.5841	0.5956	0.0002	0.0339	0.0298	3.5841
	3	0.0012	0.0032	5.1834	3.6247	0.4935	0.0191	0.0390	0.3370	3.6247
	4	0.0025	0.0066	4.4934	4.3084	0.5498	0.0203	0.0418	0.5561	4.4572
Clayton-Lognormal	1	0	0.0025	0.0495	3.0039	0.5766	0	0.0255	0.0035	3.0039
	2	0	0.0050	0.0615	3.0060	0.5748	0	0.0379	0.0033	3.0060
	3	0.0012	0.0029	2.2848	6.4614	0.7807	0.0039	0.0085	1.2022	6.4614
	4	0.0025	0.0057	2.1384	6.4003	0.7973	0.0061	0.0134	1.2021	6.4003

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Table II lists the moments of Z and $\tilde{Z}$ when $x = 2$ and $y = 2$. After power transformation with the optimal q, the kurtosis $\alpha_{4\tilde{Z}}$ matches the target κ_{target} derived from the SNT reference distribution. The skewness $\alpha_{3\tilde{Z}}$ is also significantly reduced compared to α_{3Z}, confirming that the transformation aligns $\tilde{G}$ with the SNT distributional form.

The goodness-of-fit of the entire distribution is assessed using the root-mean-square error (RMSE) (Chai and Draxler, 2014) and the Kolmogorov-Smirnov (K-S) test (Lopes et al., 2007). For all cases, the K-S test yields $H_0 = 0$ at the 1% confidence level, indicating that the BSN distribution successfully passes the test. The RMSE values for the fitted JCDFs and JPDFs are summarized in Table III, with corresponding visual comparisons shown in Figs. 1–3 (JCDFs) and Figs. 4–6 (JPDFs). Note that in each figure, subfigures (a)–(d) correspond to Cases 1–4, respectively.

Table III. Fitted RMSE values of common bivariate JCDF and JPDF.

Distribution	JCDF RMSE				JPDF RMSE			
	Case 1	Case 2	Case 3	Case 4	Case 1	Case 2	Case 3	Case 4
Normal	0.0041	0.0055	0.0104	0.0144	2.1579	1.7366	3.7504	2.9466
Weibull	0.0057	0.0070	0.0045	0.0057	3.3169	1.8982	5.3465	3.1334
Clayton-Lognormal	0.0042	0.0058	0.0103	0.0141	2.4515	1.7538	3.3419	2.6549

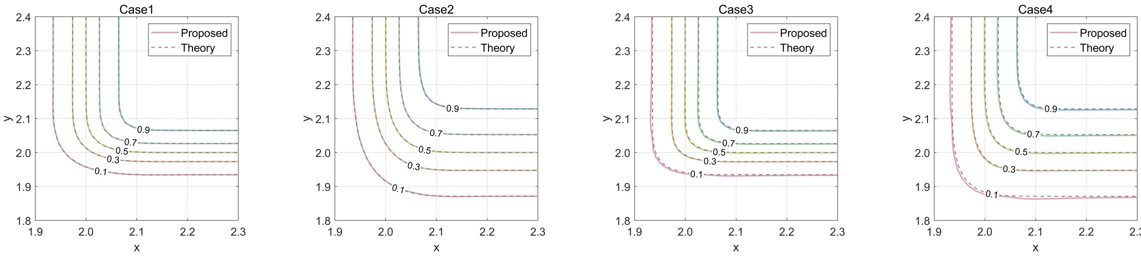

Figure 1. JCDF of bivariate Gaussian distribution.

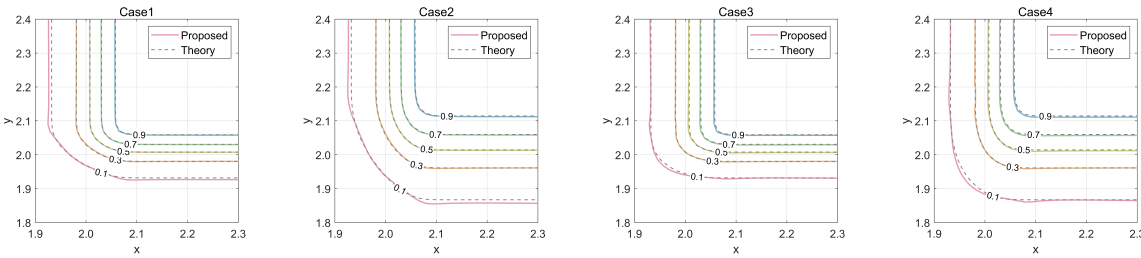

Figure 2. JCDF of bivariate Weibull distribution.

The BSN distribution achieves good JCDF agreement across all cases (RMSE < 0.015) and passes the K-S test at 1%. JPDF RMSE ranges from 1.73 to 5.35. The bivariate Weibull case shows the largest error (5.3465), reflecting the difficulty of density reconstruction for strongly non-Gaussian distributions. The BSN distribution performs acceptably for engineering applications.

4. Conclusion

This study proposes the bivariate squared normal (BSN) distribution for modeling non-Gaussian dependent variables using only the first three moments. The JCDF is constructed via probability

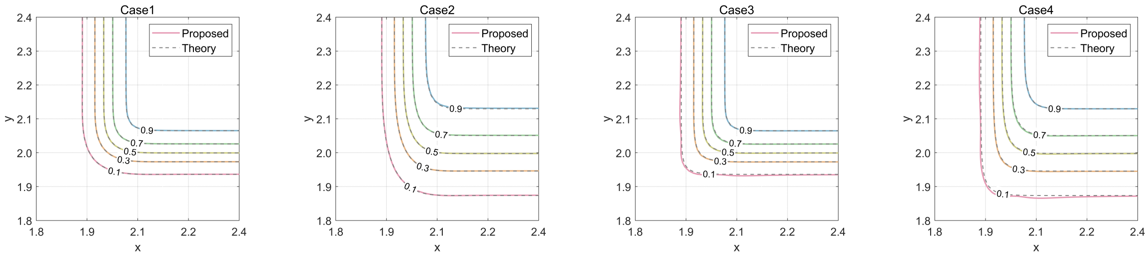

Figure 3. JCDF of bivariate distribution with Clayton copula and lognormal marginal distribution.

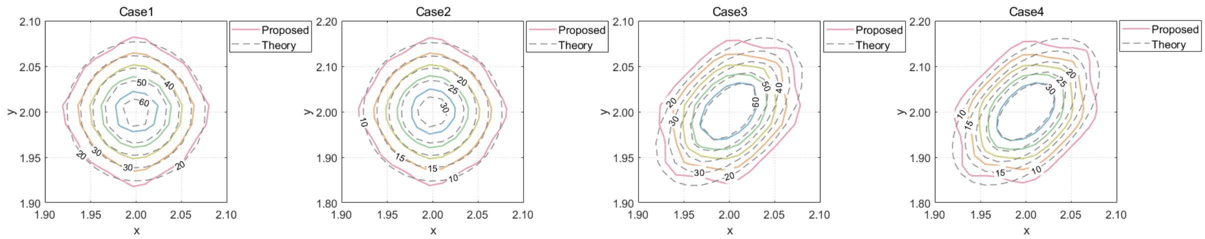

Figure 4. JPDF of bivariate Gaussian distribution.

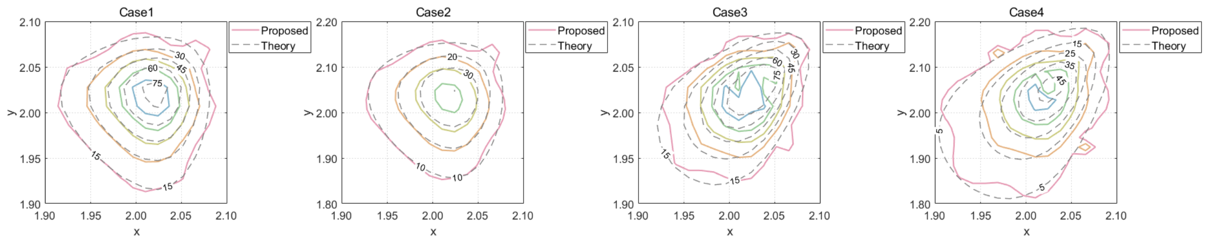

Figure 5. JPDF of bivariate Weibull distribution.

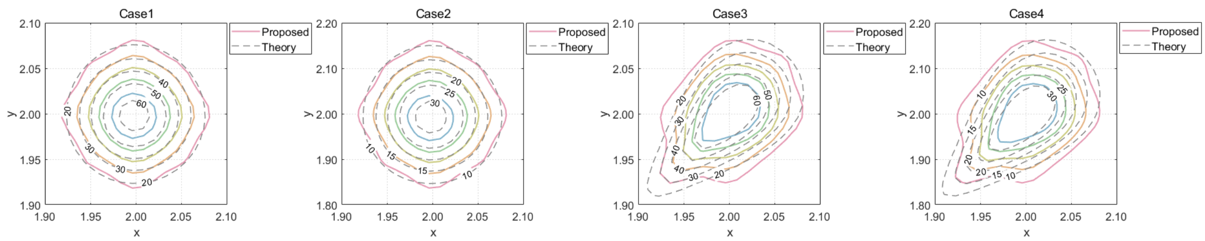

Figure 6. JPDF of bivariate distribution with Clayton copula and lognormal marginal distribution.

decomposition, with a power transformation to align kurtosis for strongly non-Gaussian cases. The JPDF is obtained by central difference.

Validation shows high JCDF accuracy (RMSE 10^{-3}–10^{-2}) and acceptable JPDF accuracy. The BSN distribution is flexible for classical and copula-based cases, offering an efficient approach for bivariate probabilistic modeling. Future work includes improving JPDF estimation and extending to higher dimensions.

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