A practical framework for failure probability assessment under aleatory and epistemic uncertainty

Pei-Pei Li[1,2*], Marcos A. Valdebenito[2], Chao Dang[2], Michael Beer[3], Matthias G.R. Faes[2]
[1] *Department of Civil Engineering, Beijing University of Technology, Beijing 100124, China*
[2] *Chair for Reliability Engineering, TU Dortmund University, Leonhard-Euler-Str. 5, Dortmund 44227, Germany*
[3] *Institute for Risk and Reliability, Leibniz University Hannover, Hannover 30167, Germany*

Abstract. In engineering applications, aleatory and epistemic uncertainties commonly coexist and interact, making their proper characterization essential for reliability analysis and informed decision-making under uncertainty. This challenge becomes particularly evident when failure probabilities of engineering structures are evaluated using incomplete, insufficient, imperfect, or imprecise information. Under such circumstances, failure probability can no longer be expressed as a single deterministic value; instead, set-theoretic or Bayesian representations are required to explicitly account for epistemic uncertainty. Although the theoretical foundations for handling these uncertainties are well established, a noticeable gap persists between academic developments and practical engineering implementation. In practice, aleatory and epistemic uncertainties,despite being conceptually distinct, are still frequently mixed, either implicitly or even explicitly in engineering analyses. To bridge this gap, this paper offers a pragmatic guide for selecting suitable uncertainty modeling frameworks, such as probability boxes, fuzzy probability models, or hierarchical probabilistic approaches, when dealing with problems influenced by both uncertainty types. By assessing the nature and quality of available information as well as the objectives of the analysis, we provide clear recommendations for appropriate modeling choices and present a comprehensive framework for evaluating failure probability. Furthermore, this work highlights the critical role of sensitivity analysis in identifying influential parameters, enabling targeted data acquisition efforts that reduce epistemic uncertainty and ultimately enhance the credibility of reliability assessments.

Keywords: Aleatory uncertainty; Epistemic uncertainty; Uncertainty modelling; Reliability analysis; Failure probability

1. Introduction

In assessing the reliability of structures, the identification and characterization of uncertainties play a critical role (Walls et al., 2017). In practice, uncertainties are commonly classified into aleatory and epistemic types (Der Kiureghian and Ditlevsen, 2009). Aleatory uncertainty represents inherent variability in physical systems, such as material properties, geometry, loads, and environmental conditions, and is therefore irreducible. In contrast, epistemic uncertainty stems from incomplete

* Corresponding author

or imperfect knowledge, including modeling assumptions, simplifications, and limited data, and can be reduced as additional information becomes available (Kiureghian, 1989). In practical engineering applications, these two types of uncertainty coexist and should be distinguished to avoid misleading interpretations (Der Kiureghian and Ditlevsen, 2009).

In an ideal scenario where complete probabilistic information is available, probability theory provides a rigorous framework for uncertainty modeling, and the failure probability can be expressed as a deterministic quantity. The failure probability can be expressed as a deterministic quantity. Let $\boldsymbol{X} = (X_1, \ldots, X_n)^T$ denote a vector of random variables and $g(\boldsymbol{X})$ the performance function, where $g(\boldsymbol{X}) \leq 0$ defines failure. The failure probability is given by

$$P_f = \int_{g(\boldsymbol{x}) \leq 0} f_{\boldsymbol{X}}(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}, \tag{1}$$

where $f_{\boldsymbol{X}}(\boldsymbol{x})$ is the joint probability density function (PDF) of $\boldsymbol{X}$.

Eq. (1) represents the fundamental problem of structural reliability analysis. It is important to point out that in the context of this equation, it is assumed that one has perfect knowledge about the full joint PDF of $\boldsymbol{X}$. However, in practical engineering applications, it is almost impossible for practical and/or economical reasons to have such perfect knowledge. This inherently introduces epistemic uncertainty into the analysis (Kiureghian, 1989). Such uncertainty may arise not only from the distribution model $f_{\boldsymbol{X}}$, but also from the limit-state function g itself (Der Kiureghian and Ditlevsen, 2009; Roy and Oberkampf, 2011). In this work, we focus on epistemic uncertainty associated with $f_{\boldsymbol{X}}$. To explicitly integrate this uncertainty in our models, a vector of variables $\boldsymbol{\theta}$ is introduced, transforming the joint PDF of $\boldsymbol{X}$ into a conditional distribution function $f_{\boldsymbol{X}}(\boldsymbol{x}|\boldsymbol{\theta})$. Accordingly, the failure probability becomes a function of $\boldsymbol{\Theta}$, referred to as the conditional failure probability:

$$P_f(\boldsymbol{\theta}) = \int_{g(\boldsymbol{x}) \leq 0} f_{\boldsymbol{X}}(\boldsymbol{x}|\boldsymbol{\theta}) \mathrm{d}\boldsymbol{x}. \tag{2}$$

Various methods have been developed to handle epistemic uncertainty. Probabilistic approaches provide a rigorous framework when sufficient data are available (Sivia and Skilling, 2006), while non-probabilistic methods such as interval models (Qiu and Elishakoff, 1998; Moens and Hanss, 2011), convex models (Ben-Haim, 1994) and fuzzy sets (Moens and Hanss, 2011; Zadeh, 1965; Hanss, 2005) are widely used in data-scarce situations. However, these approaches are primarily designed for epistemic uncertainty alone and do not explicitly account for aleatory variability (Beer et al., 2013a).

When modeling aleatory and epistemic uncertainties simultaneously, it is crucial to treat them separately to avoid conflating their distinct sources (Roy and Oberkampf, 2011). However, distinguishing between aleatory and epistemic uncertainties during the modeling phase remains challenging (Oberkampf et al., 2004; Beer et al., 2013a). To address hybrid uncertainty, several frameworks have been proposed, including probability boxes (p-boxes) (Faes et al., 2021), fuzzy probability models (Möller and Beer, 2013), and hierarchical Bayesian models (Zhao et al., 2018; Li et al., 2022).

The p-box framework is an effective tool for reliability analysis under data-scarce or incomplete-information conditions. By defining lower and upper bounds of cumulative distribution functions

(CDFs), p-boxes describe epistemic uncertainty through interval-valued failure probabilities and are widely used in engineering applications involving both aleatory and epistemic uncertainties (Faes et al., 2021; Beer et al., 2013a). However, the interval representation is often too coarse to capture the detailed influence of epistemic uncertainty on structural reliability. To provide a more flexible representation, fuzzy probability extends interval-based approaches by introducing membership functions to characterize epistemic uncertainty in probabilistic models (Möller and Beer, 2013). In this framework, the failure probability is represented as a fuzzy variable, enabling sensitivity-related analysis of the influence of uncertain parameters. Nevertheless, neither p-boxes nor fuzzy probability models possess an intrinsic mechanism for updating epistemic uncertainty when new data become available. Bayesian hierarchical probability models provide a systematic framework for uncertainty updating (Yuen, 2010). By separating aleatory and epistemic uncertainties into different probabilistic levels, these models enable the incorporation of additional observations to reduce epistemic uncertainty and improve reliability assessment.

Although these methods have been widely studied, clear guidelines for selecting appropriate approaches in practical engineering applications remain limited. Therefore, this paper aims to provide an engineering-oriented comparison of p-boxes, fuzzy probability, and hierarchical probability models for structural reliability analysis under hybrid uncertainty. The focus is placed on their uncertainty representation, interpretability, and applicability under different information conditions. The remainder of this paper is organized as follows. Section 2 introduces the theoretical frameworks of the three approaches. Section 3 discusses practical considerations in engineering applications. Section 4 presents a strip footing example for demonstration. Finally, Section 5 summarizes the main conclusions.

2. Modeling and propagation of aleatory and epistemic uncertainties

2.1. Probability box model

In some practical engineering applications, due to lack of knowledge, the probability distribution of random variables is unknown but can be constrained. In this case, both distribution-free and parametric p-boxes can be constructed based on available information (Faes et al., 2021). Within the p-box framework, the true but unknown CDF $F_{X_i}(x_i)$ of a random variable X_i is assumed to lie within the interval $[\underline{F}_{X_i}(x_i) \leq F_{X_i}(x_i) \leq \overline{F}_{X_i}(x_i) | x_i \in D_{X_i}]$ (Ferson and Ginzburg, 1996; Ferson et al., 2003), where $\underline{F}_{X_i}(x_i)$ and $\overline{F}_{X_i}(x_i)$ denote the lower and upper bounds CDF of X_i, respectively. The p-box concepts define lower probability operator $\underline{\mathbb{P}}$ and upper probability operator $\overline{\mathbb{P}}$ for events $\{X_i \leq x_i\} = (-\infty, x_i] \cap D_{X_i}$, such that $\underline{\mathbb{P}}(X_i \leq x_i) = \underline{F}_{X_i}(x_i)$ and $\overline{\mathbb{P}}(X_i \leq x_i) = \overline{F}_{X_i}(x_i)$. This formulation generates a credal set of probability measures. The p-box framework effectively handles the epistemic uncertainty, such as that caused by incomplete information on $F_{X_i}(x_i)$, by assigning an interval $[\underline{F}_{X_i}(x_i), \overline{F}_{X_i}(x_i)]$ for each value of X_i (Ferson et al., 2003). If high-quality information is available across the full range of possible values for X_i this interval becomes narrow, and the p-box approximates a crisp, deterministic CDF. However, when less information is available, the bounds widen to reflect weaker confidence in the resulting estimates.

When no prior information on the distribution family is available, distribution-free p-boxes can be adopted (Faes et al., 2022). Such models are constructed solely from the lower and upper bounds of the CDF, where any admissible function that is non-decreasing, right-continuous, and bounded within $[0, 1]$ can represent the true CDF. Owing to this flexibility, distribution-free p-boxes provide a highly general framework for modeling both aleatory and epistemic uncertainties. If the distribution family is known while the distribution parameters $\boldsymbol{\theta}$ are interval-valued, parametric p-boxes are more appropriate. In this framework, admissible CDFs are constrained through bounded parameter domains, allowing a clear separation between aleatory uncertainty and epistemic uncertainty. Specifically, the distribution family characterizes aleatory variability, whereas the parameter intervals represent epistemic uncertainty (Zhang et al., 2013). The lower and upper bounds of the parametric p-box are obtained by optimizing the conditional CDF over the parameter domain:

$$\underline{F}_X(x) = \min_{\boldsymbol{\theta} \in D_{\boldsymbol{\theta}}} F_X(x \mid \boldsymbol{\theta}), \tag{3}$$

$$\overline{F}_X(x) = \max_{\boldsymbol{\theta} \in D_{\boldsymbol{\theta}}} F_X(x \mid \boldsymbol{\theta}), \tag{4}$$

where $F_X(x \mid \boldsymbol{\theta})$ denotes the conditional CDF and $D_{\boldsymbol{\theta}}$ represents the domain of the parameter vector $\boldsymbol{\theta}$.

When p-boxes are used as inputs in reliability analysis, the failure probability is no longer a single deterministic value, but an interval-valued quantity $[\underline{P}_f, \overline{P}_f]$. The lower and upper bounds of the failure probability can be expressed in a unified form as (Alvarez et al., 2014; Schöbi and Sudret, 2017)

$$\underline{P}_f = \mathbb{P}\left(\max_{\boldsymbol{X} \in [\underline{\boldsymbol{X}}, \overline{\boldsymbol{X}}]} g(\boldsymbol{X}) \leq 0\right), \tag{5}$$

$$\overline{P}_f = \mathbb{P}\left(\min_{\boldsymbol{X} \in [\underline{\boldsymbol{X}}, \overline{\boldsymbol{X}}]} g(\boldsymbol{X}) \leq 0\right), \tag{6}$$

where $\mathbb{P}(\cdot)$ denotes the probability operator and $g(\boldsymbol{X})$ is the limit-state function. The width of the failure probability interval directly reflects the level of epistemic uncertainty: wider intervals indicate larger uncertainty in the reliability estimate, whereas improved information quality generally leads to narrower intervals approaching a crisp value. In engineering practice, the upper bound of the failure probability corresponds to the worst-case scenario. However, relying solely on the upper bound in structural design may lead to overly conservative solutions. Therefore, the entire failure probability interval should be considered to achieve a balance between safety and cost-effectiveness.

Furthermore, identifying the uncertain parameters that dominate the failure probability is essential for guiding data collection and reducing epistemic uncertainty. Traditional sensitivity analysis methods are generally difficult to extend to interval-valued failure probabilities. In contrast, fuzzy probability methods provide a global description of uncertainty propagation and help identify influential parameters, while Bayesian approaches further refine probability estimates through data updating. The combination of these methods offers a comprehensive framework for epistemic uncertainty quantification, data acquisition, and reliability-based structural design.

2.2. Fuzzy probability model

When precise bounds in interval models cannot be strictly specified, fuzzy set theory provides an effective framework for representing wide-domain uncertainty through gradually varying membership functions (Zadeh, 1965). In this context, epistemic uncertainty in the distribution parameters $\boldsymbol{\Theta}$ is modeled using fuzzy sets. The fuzzy set associated with the l-th distribution parameter of the i-th random variable is defined as

$$\widetilde{\theta}_{i,l} = \left\{ (\theta_{i,l}, \mu_{\widetilde{\theta}_{i,l}}(\theta_{i,l})) \,\middle|\, \theta_{i,l} \in \Theta_{i,l},\ \mu_{\widetilde{\theta}_{i,l}}(\theta_{i,l}) \in (0,1] \right\}, \tag{7}$$

where $\mu_{\widetilde{\theta}_{i,l}}(\theta_{i,l})$ represents the membership function that assigns a degree of membership to each value $\theta_{i,l}$. The value of $\mu_{\widetilde{\theta}_{i,l}}(\theta_{i,l})$ reflects how strongly $\theta_{i,l}$ belongs to the fuzzy set. A membership value closer to 1 indicates a higher degree of confidence.

Membership functions can represent similarity, preference, or uncertainty (Dubois and Prade, 1997; Möller and Beer, 2008). In this study, they are interpreted as a representation of epistemic uncertainty for structural reliability analysis. Fuzzy operations are carried out using the minimum–maximum operators and the extension principle (Zimmermann, 2011). To improve computational efficiency, the α-cut discretization method is adopted instead of directly applying the extension principle (Möller et al., 2000). This approach transforms a fuzzy set into a series of interval sets at different α-levels, while preserving the essential uncertainty information. For simplicity, triangular membership functions are used to describe uncertain distribution parameters. Under convex membership assumptions, each α-level corresponds to an interval representation, allowing the fuzzy set to be expressed as a family of nested intervals indexed by α (Möller and Beer, 2013).

Within this framework, aleatory uncertainty is modeled by probability distributions, while epistemic uncertainty in distribution parameters is described using fuzzy sets, leading to a fuzzy probability formulation (Beer et al., 2013b). Accordingly, the random variable is extended to a fuzzy random variable $\widetilde{X}_i$, whose fuzzy CDF is defined as

$$\tilde{F}_{X_i}(x_i|\boldsymbol{\theta}_i) = \left\{ \big(F_{X_i}(x_i|\boldsymbol{\theta}_i), \mu\big(F_{X_i}(x_i|\boldsymbol{\theta}_i)\big)\big) \,\middle|\, F_{X_i}(x_i|\boldsymbol{\theta}_i) \in \mathbb{F},\ \mu\big(F_{X_i}(x_i|\boldsymbol{\theta}_i)\big) \in (0,1] \right\}, \tag{8}$$

where $\mathbb{F}$ denotes the admissible set of CDFs. For a given α-level, the corresponding α-cut of the fuzzy CDF can be expressed as

$$\tilde{F}^{\alpha}_{X_i}(x_i|\boldsymbol{\theta}_i) = \left\{ F_{X_i}(x_i|\boldsymbol{\theta}_i) \in \mathbb{F} \,\middle|\, \mu\big(F_{X_i}(x_i|\boldsymbol{\theta}_i)\big) \geq \alpha \right\}. \tag{9}$$

For each α-level, the lower and upper bounds of the fuzzy CDF are obtained through optimization over the parameter domain:

$$\underline{F}^{\alpha}_{X_i}(x_i) = \min_{\boldsymbol{\theta}_i \in D_{\boldsymbol{\theta}_i}} \tilde{F}^{\alpha}_{X_i}(x_i|\boldsymbol{\theta}_i), \tag{10}$$

$$\overline{F}^{\alpha}_{X_i}(x_i) = \max_{\boldsymbol{\theta}_i \in D_{\boldsymbol{\theta}_i}} \tilde{F}^{\alpha}_{X_i}(x_i|\boldsymbol{\theta}_i), \tag{11}$$

where $D_{\boldsymbol{\theta}_i}$ denotes the domain of the distribution parameters.

Due to the fuzziness of the probabilistic model, the structural failure probability also becomes a fuzzy variable (Beer, 2004; Valdebenito et al., 2020). The membership function of the failure probability can be constructed using the α-level optimization approach (Hanss, 2005; Beer, 2004). At each α-level, the fuzzy parameters are transformed into interval vectors $\boldsymbol{\theta}_{\alpha}^{I}$, and the lower and upper bounds of the failure probability are obtained as

$$\underline{P}_f^{\alpha} = \min_{\boldsymbol{\theta} \in \boldsymbol{\theta}_{\alpha}^{I}} P_f(\boldsymbol{\theta}), \tag{12}$$

$$\overline{P}_f^{\alpha} = \max_{\boldsymbol{\theta} \in \boldsymbol{\theta}_{\alpha}^{I}} P_f(\boldsymbol{\theta}). \tag{13}$$

By performing analyses at multiple discrete α-levels, a nested family of failure probability intervals is obtained, from which the membership function of the failure probability can be reconstructed. Since each α-cut of fuzzy parameters is equivalent to an interval parameter set, fuzzy probability analysis can be interpreted as a sequence of p-box analyses across different α-levels. This framework captures the coupled effects of aleatory and epistemic uncertainties. It also reflects sensitivity to epistemic uncertainty in distribution parameters. In general, higher membership levels indicate reduced epistemic uncertainty and narrower failure probability intervals, while significant interval expansion under small parameter perturbations implies strong sensitivity to modeling uncertainty.

However, fuzzy probability primarily captures the overall sensitivity of the failure probability with respect to epistemic uncertainty, but does not explicitly identify the dominant uncertain parameters in $\boldsymbol{\Theta}$. Therefore, dedicated sensitivity analysis methods under fuzzy uncertainty are required to quantify the contribution of individual parameters and support robust design. Although existing methods have made progress, further development is still needed to better characterize epistemic uncertainty propagation. In this context, fuzzy numbers, as a class of random sets, provide a useful theoretical basis for uncertainty propagation and reliability-based design.

2.3. Hierarchical probability model

When sufficient expert knowledge, prior assumptions, and measurement data are available, epistemic uncertainty can be modeled within a probabilistic framework. In this case, both aleatory and epistemic uncertainties can be consistently treated using a hierarchical probability model (Li et al., 2022). A key distinction is that aleatory uncertainty describes inherent randomness in structural systems, whereas epistemic uncertainty reflects incomplete knowledge of model parameters.

In the hierarchical probability model, the distribution parameters $\boldsymbol{\theta}$ are treated as random variables in the outer loop, while aleatory uncertainty is represented by the random variable $\boldsymbol{X}$ with conditional PDF $f_{\boldsymbol{X}}(\boldsymbol{x}|\boldsymbol{\theta})$ in the inner loop. Consequently, the failure probability also becomes a random variable.

The expected failure probability P_F can be estimated using MCS as

$$P_F = \int_{\boldsymbol{\theta}} P_f(\boldsymbol{\theta}) f_{\boldsymbol{\Theta}}(\boldsymbol{\theta}) d\boldsymbol{\theta} \approx \frac{1}{M} \sum_{j=1}^{M} P_f(\boldsymbol{\theta}_j), \tag{14}$$

where M denotes the number of samples.

In addition to Monte Carlo simulation, several alternative approaches have been developed to estimate the expected failure probability. For example, the point estimation method (Hong, 1996) is used to evaluate the mean failure probability, while Li et al. (Li et al., 2022) derives it based on the distribution of reliability indices. Similarly, Zhao et al. (Zhao et al., 2018) and Li et al. (Li et al., 2022) obtain quantiles of the failure probability from the reliability index distribution. Although statistical measures such as the mean, quantiles, and confidence intervals are useful for decision-making under uncertainty, they inherently couple aleatory and epistemic uncertainties, making it difficult to isolate epistemic effects. Therefore, obtaining the full distribution (PDF/CDF) of the failure probability is essential. This allows a more complete characterization of uncertainty and enables explicit analysis of how epistemic uncertainty influences the variability of failure probability.

Several studies have addressed this issue from different perspectives. A first-order approximation (Der Kiureghian and Ditlevsen, 2009) is used to construct an explicit distribution of the failure probability, while Li et al. (Li et al., 2022) proposes an analytical formulation based on moment methods. These approaches incorporate epistemic uncertainty into the probabilistic model and partially decouple aleatory and epistemic uncertainties. Once the failure probability distribution is available, its coefficient of variation and confidence intervals can be used to quantify epistemic uncertainty. Larger variability indicates stronger epistemic uncertainty. It is also noted that extreme quantiles may lead to failure probabilities approaching zero, which limits their practical interpretability. When uncertainty bounds are excessively wide, global reliability sensitivity measures (Cui et al., 2010; Wei et al., 2012) can be used to identify influential parameters and guide targeted uncertainty reduction.

To further reduce epistemic uncertainty, Bayesian updating can be introduced to refine parameter distributions using new observed data D_X. The posterior distribution of the uncertain parameters is given by

$$f'_{\boldsymbol{\Theta}}(\boldsymbol{\theta}|D_{\boldsymbol{X}}) = k\, L(D_{\boldsymbol{X}}|\boldsymbol{\theta})\, f_{\boldsymbol{\Theta}}(\boldsymbol{\theta}), \tag{15}$$

where the normalization constant is

$$k = \left[\int_{-\infty}^{\infty} L(D_{\boldsymbol{X}}|\boldsymbol{\theta}) f_{\boldsymbol{\Theta}}(\boldsymbol{\theta})\, d\boldsymbol{\theta}\right]^{-1}. \tag{16}$$

The likelihood function is defined as

$$L(D_{\boldsymbol{X}}|\boldsymbol{\theta}) \propto \mathbb{P}(D_{\boldsymbol{X}}|\boldsymbol{\Theta} = \boldsymbol{\theta}), \tag{17}$$

and under the assumption of independent observations, it can be written as

$$L(D_{\boldsymbol{X}}|\boldsymbol{\theta}) = \prod_{i=1}^{N} f_{\boldsymbol{X}|\boldsymbol{\Theta}}(D_{\boldsymbol{X}_i}|\boldsymbol{\theta}). \tag{18}$$

The likelihood function measures the agreement between model parameters and observed data, and serves as the key link between prior and posterior distributions. When prior knowledge is well supported by both theory and data, Bayesian updating effectively reduces epistemic uncertainty.

In this framework, updating is typically performed on distribution parameters, which are then propagated to update the failure probability. An alternative strategy directly updates the random variables; however, this approach may lead to a deterministic failure probability, which obscures the distinction between inherent randomness and epistemic uncertainty and weakens the physical interpretation of the updating process.

In contrast, Bayesian inference provides posterior distributions and associated credible intervals, offering a quantitative measure of epistemic uncertainty. As more data are incorporated, the posterior distribution becomes progressively refined, leading to a reduction in both the coefficient of variation and credible intervals of the failure probability. This iterative updating process enables systematic control of epistemic uncertainty and provides a consistent and robust framework for reliability assessment and engineering decision-making.

3. Practical notes

In structural reliability analysis, both aleatory and epistemic uncertainties must be properly considered. The selection of an appropriate modeling approach depends on the available information and the intended engineering objective. The former determines the degree of uncertainty that can be quantified, while the latter defines the required decision support.

These methods represent different levels of epistemic uncertainty modeling capability, ranging from interval-based to fully probabilistic representations. Classical reliability analysis assumes fully specified probability distributions and focuses on aleatory uncertainty, while epistemic uncertainty is not explicitly addressed.

For limited information, the p-box framework provides upper and lower bounds of the cumulative distribution function, leading to interval-valued failure probabilities. This makes it particularly suitable for safety-oriented assessment problems. However, overly wide bounds may limit its usefulness in design optimization and sensitivity analysis. Fuzzy probability models provide a more flexible representation of epistemic uncertainty by describing its influence on failure probability in a graded manner. Based on the sensitivity of the resulting bounds, one can judge whether additional data collection and Bayesian updating are required. In contrast, hierarchical Bayesian models treat distribution parameters as random variables and enable systematic updating as new data become available, progressively reducing epistemic uncertainty. However, their ability to explicitly represent worst-case scenarios is more limited compared with interval-based approaches.

Overall, these methods constitute a complementary framework for hybrid uncertainty modeling. The choice of method should be guided by the available information and the specific engineering objective. In practice, combining multiple methods with sensitivity analysis is often beneficial for identifying influential uncertain parameters and improving the robustness of engineering decisions.

4. Application example

This section presents a classical strip footing problem from Sudret (2014) (Sudret, 2014) to illustrate the application of each method investigated in this study. The strip footing has a width of $B =10$

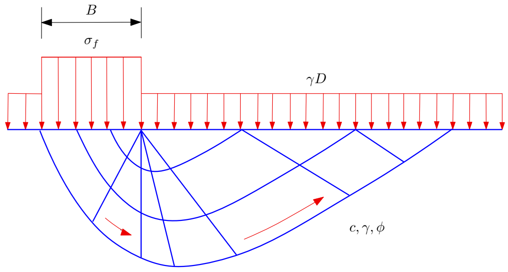


Figure 1. Strip footing

Table I. Probabilistic information for the random variables

Method Parameter	Distribution	Mean value	CoV
B	Deterministic	10 m	–
D	Gaussian (truncated)	1 m	0.15
γ	Lognormal	20 kN/m^3	0.1
c	Lognormal	20 kPa	0.25
ϕ	Beta	Range: [0, 45]$^\circ$, $\mu_\phi = 30^\circ$	$\approx$0.1

m and is embedded at a depth D, as illustrated in Fig. 1. It is assumed that groundwater is located far below the surface, and the soil is homogeneous with cohesion c, friction angle ϕ and unit weight γ. The limit-state function is defined as the ultimate bearing capacity not exceeding its design value (Lang et al., 2011):

$$g(\boldsymbol{X}) = q_u(\boldsymbol{X}) - q_{des} = cN_c + \gamma D N_q + \frac{1}{2} B\gamma N_\gamma - q_{des}, \tag{19}$$

where $N_q = e^{\pi \tan\phi} \tan^2\left(\frac{\pi}{4} + \frac{\phi}{2}\right)$, $N_c = (N_q - 1)\cot\phi$, and $N_\gamma = 2(N_q - 1)\tan\phi$.

The variables D, γ, c, and ϕ are treated as independent random variables with properties listed in Table I. The design value q_{des} is obtained by dividing the ultimate bearing capacity by the safety factor (SF), i.e., q_{des}=$\overline{q_u}$/SF. The value of $\overline{q_u}$ is determined using the mean values of the parameters listed in Table I. With a safety factor of 2, the resulting design value is q_{des}= 2.78 MPa.

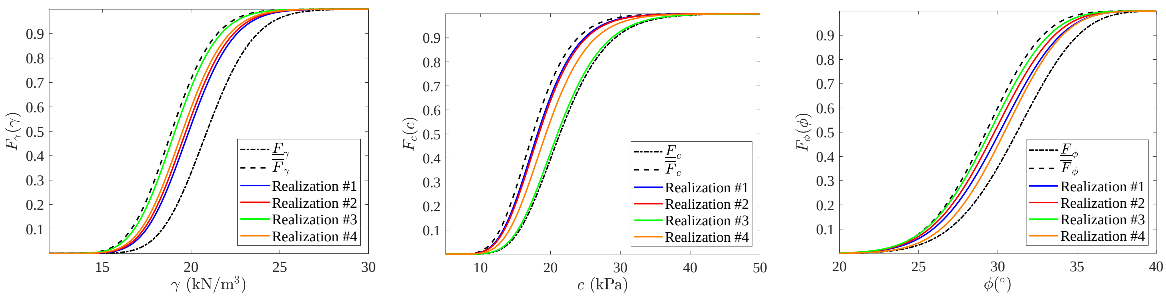
Considering only aleatory uncertainty, Monte Carlo simulation (MCS) with 10^6 samples gives the failure probability as $\hat{P}_f = 3.9345 \times 10^{-2}$. Due to limited information, there is significant epistemic uncertainty in estimating the mean values of γ and c (i.e., μ_γ and μ_c), as well as the distribution parameters of ϕ (i.e., α_ϕ and β_ϕ). It is assumed that the interval values of μ_γ, μ_c, α_ϕ and β_ϕ are given as follows: $\mu_\gamma \in [19, 21]$ kN/m^3, $\mu_c \in [18, 22]$ kPa, $\alpha_\phi \in [29, 35]$, and $\beta_\phi \in [13, 19]$.

Given the known distribution families (Table I), parametric p-boxes are used to model the CDFs of γ, c, and ϕ, as given in Fig. 2. By combining particle swarm optimization (PSO) (Kennedy and

Eberhart, 1995) with MCS, the lower and upper bounds of the failure probability are estimated as

$$\underline{\hat{P}}_f = \mathbb{P}\left(\max_{X_i \in [\underline{X}_i, \overline{X}_i]} g(\boldsymbol{X}) \leq 0 \right) = 2.769 \times 10^{-3}, \tag{20}$$

$$\overline{\hat{P}}_f = \mathbb{P}\left(\min_{X_i \in [\underline{X}_i, \overline{X}_i]} g(\boldsymbol{X}) \leq 0 \right) = 2.243 \times 10^{-1}. \tag{21}$$



(a) Soil weight γ (b) Cohesion c (c) Friction angle ϕ°

Figure 2. Parametric p-box of γ, c, and ϕ

As shown in Eqs. 20–21, the failure probability spans nearly two orders of magnitude, indicating a strong influence of epistemic uncertainty. To further investigate this effect, a fuzzy probability analysis is performed using triangular membership functions for μ_γ, μ_c, α_ϕ, and β_ϕ. The resulting fuzzy failure probability is obtained based on Eqs. 12–13. At membership level 1, the failure probability is $\hat{P}_f = 3.9172 \times 10^{-2}$, which is close to the case without epistemic uncertainty. At membership level 0, the failure probability varies within $[2.866 \times 10^{-3}, 2.226 \times 10^{-1}]$, consistent with the parametric p-box bounds. The obtained membership function shows that the failure probability is highly sensitive to epistemic uncertainty, suggesting that additional information could significantly reduce uncertainty.

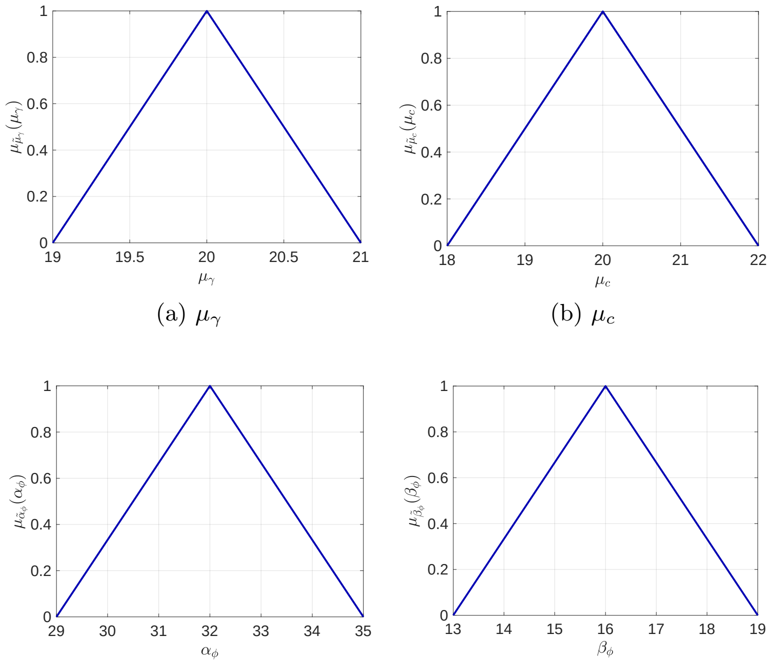
Motivated by this observation, a hierarchical Bayesian model is further adopted. Prior distributions are assigned to μ_γ, μ_c, α_ϕ, and β_ϕ as listed in Table II. Using a double-loop MCS, the expected failure probability is estimated as 5.345×10^{-2} with a coefficient of variation of 0.750. The corresponding credible intervals are 95% $[7.441 \times 10^{-3}, 1.585 \times 10^{-1}]$, 90% $[1.019 \times 10^{-2}, 1.317 \times 10^{-1}]$, and 75% $[1.631 \times 10^{-2}, 9.692 \times 10^{-2}]$.

Furthermore, additional soil property data reported by Raviteja et al. (Raviteja and Basha, 2021) are used to update the model parameters. The dataset includes soil unit weight γ, cohesion c, and friction angle ϕ, denoted as

$$[D_\gamma, D_c, D_\phi] = \begin{bmatrix} 14.1 & 23 & 24 \\ 17.6 & 29 & 39 \\ 15.9 & 28 & 32 \\ 12.5 & 30 & 37 \\ 13.2 & 18 & 29.2 \end{bmatrix}.$$

Table II. Prior distribution of the distribution parameters

Parameter	Distribution	Mean value	CoV
μ_γ	Lognormal	20 kN/m^3	0.075
μ_c	Lognormal	20 kPa	0.075
α_ϕ	Lognormal	32	0.075
β_ϕ	Lognormal	16	0.075



(a) μ_γ (b) μ_c

(c) α_ϕ (d) β_ϕ

Figure 3. Membership functions of μ_γ, μ_c, α_ϕ, and β_ϕ

Based on these observations, Bayesian inference is applied to update the distributions of the model parameters. Combining Eq. 2 and Eq. 15, the prior and posterior failure probability distributions are obtained using Markov chain Monte Carlo (MCMC) coupled with Monte Carlo simulation with $M = 10^6$ samples (see Fig. 4).

The results indicate that the CoV of the failure probability serves as an effective measure of epistemic uncertainty. After Bayesian updating, the CoV of the failure probability is reduced from 0.750 to 0.383, indicating a 49% reduction in relative epistemic uncertainty. The credibility intervals for the failure probability have been updated as follows: the 95% credibility interval is $[1.074 \times 10^{-1}, 5.471 \times 10^{-1}]$; the 90% credibility interval is $[1.295 \times 10^{-1}, 5.034 \times 10^{-1}]$; and the 75% credibility

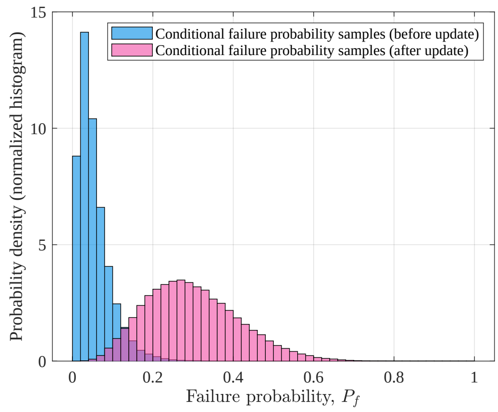


Figure 4. Histogram of the failure probability before and after update

interval is $[1.694 \times 10^{-1}, 4.343 \times 10^{-1}]$. As additional measurements become available, both the CoV and credibility intervals are expected to further decrease, leading to more stable and reliable failure probability estimates. This refined uncertainty quantification framework provides a rational basis for geotechnical reliability design under both aleatory and epistemic uncertainties.

5. Summary and Conclusions

This paper investigated structural reliability analysis under both aleatory and epistemic uncertainties from an engineering perspective. While classical probabilistic methods effectively handle aleatory uncertainty, they are often insufficient when epistemic uncertainty is significant due to incomplete information.

Three representative approaches for hybrid uncertainty modeling are compared: p-boxes, fuzzy probability, and hierarchical Bayesian models. The p-box approach provides efficient and conservative bounds for failure probability, making it suitable for safety-critical applications, but its rigid interval representation limits detailed sensitivity interpretation. The fuzzy probability model extends this framework by capturing the effect of parameter variations, offering more informative bounds for uncertainty propagation and reliability assessment. In contrast, the hierarchical Bayesian model reduces epistemic uncertainty through data assimilation and iterative updating, allowing failure probability estimates to converge as more information becomes available.

Sensitivity analysis plays a key role in identifying influential parameters and guiding targeted data collection to reduce epistemic uncertainty. However, global sensitivity analysis methods that simultaneously account for aleatory and epistemic uncertainties remain challenging.

In summary, method selection depends on data availability and analysis objectives: p-boxes for conservative assessment, fuzzy probability for uncertainty interpretation, and Bayesian models for data-driven updating. Future work should further improve model integration and sensitivity analysis methods to support complex engineering applications.

Acknowledgements

The authors gratefully acknowledge the support of the Alexander von Humboldt Foundation for the postdoctoral grant of Pei-Pei Li, the Henriette Herz Scouting program (Matthias G.R. Faes), and the financial support of the German Research Foundation (DFG) for Chao Dang (Grant number 530326817).

References

Alvarez, D. A., J. E. Hurtado, and F. Uribe: 2014, 'Estimation of the lower and upper probabilities of failure using random sets and subset simulation'. In: *Vulnerability, Uncertainty, and Risk: Quantification, Mitigation, and Management*. pp. 905–914.

Beer, M.: 2004, 'Uncertain structural design based on nonlinear fuzzy analysis'. *ZAMM-Journal of Applied Mathematics and Mechanics/Zeitschrift für Angewandte Mathematik und Mechanik: Applied Mathematics and Mechanics* **84**(10-11), 740–753.

Beer, M., S. Ferson, and V. Kreinovich: 2013a, 'Imprecise probabilities in engineering analyses'. *Mechanical Systems and Signal Processing* **37**(1–2), 4–29.

Beer, M., Y. Zhang, S. T. Quek, and K. K. Phoon: 2013b, 'Reliability analysis with scarce information: Comparing alternative approaches in a geotechnical engineering context'. *Structural Safety* **41**, 1–10.

Ben-Haim, Y.: 1994, 'Convex models of uncertainty: applications and implications'. *Erkenntnis* **41**(2), 139–156.

Cui, L., Z. Lü, and X. Zhao: 2010, 'Moment-independent importance measure of basic random variable and its probability density evolution solution'. *Science China Technological Sciences* **53**, 1138–1145.

Der Kiureghian, A. and O. Ditlevsen: 2009, 'Aleatory or epistemic? Does it matter?'. *Structural Safety* **31**(2), 105–112.

Dubois, D. and H. Prade: 1997, 'The three semantics of fuzzy sets'. *Fuzzy sets and systems* **90**(2), 141–150.

Faes, M. G., M. Broggi, G. Chen, K.-K. Phoon, and M. Beer: 2022, 'Distribution-free P-box processes based on translation theory: Definition and simulation'. *Probabilistic Engineering Mechanics* **69**, 103287.

Faes, M. G., M. Daub, S. Marelli, E. Patelli, and M. Beer: 2021, 'Engineering analysis with probability boxes: A review on computational methods'. *Structural Safety* **93**, 102092.

Ferson, S. and L. R. Ginzburg: 1996, 'Different methods are needed to propagate ignorance and variability'. *Reliability Engineering & System Safety* **54**(2-3), 133–144.

Ferson, S., V. Kreinovich, L. Ginzburg, D. S. Myers, and K. Sentz: 2003, *Constructing probability boxes and Dempster-Shafer structures*, No. 4015. Sandia National Laboratories.

Hanss, M.: 2005, *Applied fuzzy arithmetic*. Springer.

Hong, H. P.: 1996, 'Evaluation of the probability of failure with uncertain distribution parameters'. *Civil Engineering Systems* **13**(2), 157–168.

Kennedy, J. and R. Eberhart: 1995, 'Particle swarm optimization'. In: *Proceedings of ICNN'95-international conference on neural networks*, Vol. 4. pp. 1942–1948.

Kiureghian, A. D.: 1989, 'Measures of structural safety under imperfect states of knowledge'. *Journal of Structural Engineering* **115**(5), 1119–1140.

Lang, Am, Amann, H.-j, Huder, J., P., Puzrin, J. Huder, and H.-J. Lang: 2011, *Bodenmechanik und grundbau*. Springer.

Li, P.-P., Y.-G. Zhao, and Z. Zhao: 2022, 'Efficient method for fully quantifying the uncertainty of failure probability'. *Computer Methods in Applied Mechanics and Engineering* **399**, 115345.

Moens, D. and M. Hanss: 2011, 'Non-probabilistic finite element analysis for parametric uncertainty treatment in applied mechanics: Recent advances'. *Finite Elements in Analysis and Design* **47**(1), 4–16.

Möller, B. and M. Beer: 2008, 'Engineering computation under uncertainty–capabilities of non-traditional models'. *Computers & Structures* **86**(10), 1024–1041.

Möller, B. and M. Beer: 2013, *Fuzzy randomness: uncertainty in civil engineering and computational mechanics*. Springer Science & Business Media.

Möller, B., W. Graf, and M. Beer: 2000, 'Fuzzy structural analysis using α-level optimization'. *Computational Mechanics* **26**(6), 547–565.

Oberkampf, W. L., J. C. Helton, C. A. Joslyn, S. F. Wojtkiewicz, and S. Ferson: 2004, 'Challenge problems: uncertainty in system response given uncertain parameters'. *Reliability Engineering & System Safety* **85**(1-3), 11–19.

Qiu, Z. and I. Elishakoff: 1998, 'Antioptimization of structures with large uncertain-but-non-random parameters via interval analysis'. *Computer Methods in Applied Mechanics and Engineering* **152**(3-4), 361–372.

Raviteja, K. and B. M. Basha: 2021, 'Characterization of variability of unit weight and shear parameters of municipal solid waste'. *Journal of Hazardous, Toxic, and Radioactive Waste* **25**(2), 04020077.

Roy, C. J. and W. L. Oberkampf: 2011, 'A comprehensive framework for verification, validation, and uncertainty quantification in scientific computing'. *Computer Methods in Applied Mechanics and Engineering* **200**(25-28), 2131–2144.

Schöbi, R. and B. Sudret: 2017, 'Structural reliability analysis for p-boxes using multi-level meta-models'. *Probabilistic Engineering Mechanics* **48**, 27–38.

Sivia, D. and J. Skilling: 2006, *Data analysis: a Bayesian tutorial*. OUP Oxford.

Sudret, B.: 2014, 'Polynomial chaos expansions and stochastic finite element methods'. *Risk and reliability in geotechnical engineering* pp. 265–300.

Valdebenito, M. A., M. Beer, H. A. Jensen, J. Chen, and P. Wei: 2020, 'Fuzzy failure probability estimation applying intervening variables'. *Structural Safety* **83**, 101909.

Walls, L., T. Bedford, M. Revie, and E. Zio: 2017, 'Reliability, risk and safety: analytical support for decision making'. *EURO Journal on Decision Processes* **5**, 1–4.

Wei, P., Z. Lu, W. Hao, J. Feng, and B. Wang: 2012, 'Efficient sampling methods for global reliability sensitivity analysis'. *Computer Physics Communications* **183**(8), 1728–1743.

Yuen, K.-V.: 2010, *Bayesian methods for structural dynamics and civil engineering*. John Wiley & Sons.

Zadeh, L. A.: 1965, 'Information and control'. *Fuzzy Sets* **8**(3), 338–353.

Zhang, H., H. Dai, M. Beer, and W. Wang: 2013, 'Structural reliability analysis on the basis of small samples: An interval quasi-Monte Carlo method'. *Mechanical Systems and Signal Processing* **37**(1–2), 137–151.

Zhao, Y.-G., P.-P. Li, and Z.-H. Lu: 2018, 'Efficient evaluation of structural reliability under imperfect knowledge about probability distributions'. *Reliability Engineering & System Safety* **175**, 160–170.

Zimmermann, H.-J.: 2011, *Fuzzy set theory—and its applications*. Springer Science & Business Media.