A Hybrid Interval Uncertainty Algorithm for Structural Analysis

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Abstract: A variety of methods have been proposed for analyzing structures with interval parameters, each with distinct strengths and limitations depending on the problem characteristics. In the context of structural analysis, intrinsic methods typically involve directly solving the governing interval equations using algorithms that preserve the interval nature of the parameters, without relying on transformations or approximations. These methods are generally efficient and provide guaranteed solution bounds for response quantities. Still, a common challenge arises when the parameterization of a few parameters is nonlinear, such as when the loading direction is unknown but bounded. By contrast, extrinsic methods reformulate the interval problem as an optimization problem, where external optimization techniques are applied to identify the extreme responses within the prescribed interval ranges. While extrinsic approaches often provide an interval solution within the interval constraints, they are computationally demanding and do not provide a guaranteed bound on a solution.

This paper explores a hybrid interval approach that integrates both intrinsic and extrinsic strategies to solve systems of linear interval equations more effectively. The hybrid framework applies intrinsic methods where computational efficiency is critical and employs extrinsic optimization where the interval parameterization is nonlinear, thereby exploiting the complementary advantages of each method. Example calculations are presented to compare the performance of the hybrid approach against conventional extrinsic optimization-based methods. The results demonstrate that hybrid methods achieve more efficient computations and deliver solutions closer to the correct bounds than extrinsic calculations alone, making them a practical and robust tool for interval-based structural analysis.

Keywords: Hybrid interval, interval, finite element, optimization

1. Introduction

Structural systems are often subject to uncertainties in material properties, geometry, and external loading. When such uncertainties are represented as bounded intervals rather than probabilistic distributions, the resulting problem formulation requires specialized solution strategies. Over the past decades, several classes of algorithms have been developed for solving structural analysis problems with interval parameters, each offering unique strengths and limitations. The choice of method depends not only on the algorithm's computational properties but also on the characteristics of the specific problem under consideration.

Optimization-based methods provide an extrinsic approach to interval analysis by recasting the problem as an optimization task over the bounded domain defined by the intervals. These methods are highly flexible, capable of handling both linear and nonlinear parameterizations, and are particularly well-suited for situations where the governing equations cannot be directly expressed in closed form. However, this generality comes at the cost of high computational demand, especially for large-scale systems with many uncertain parameters.

In contrast, intrinsic methods such as fixed-point algorithms exploit the algebraic structure of the interval equations directly. They are typically more computationally efficient and can guarantee enclosure bounds for key response quantities such as nodal displacements, element forces, stresses, and strains. Another important class of intrinsic methods, sensitivity end-point techniques, which take advantage of the end-point property, where the solution depends only on combinations of parameter extremes. These methods can be both fast and accurate, but their applicability is restricted to problems that possess the end-point property, such as linear affine combinations of rank-one matrices.

Despite these advances, a persistent challenge arises when the parameterization is nonlinear. A typical example is when the direction of an external load is uncertain but constrained within a bounded interval. In such cases, even if most of the interval parameters could be treated more efficiently with intrinsic methods, the entire problem is often forced into an extrinsic optimization framework due to the presence of a single nonlinear parameterization. This mismatch between problem structure and method applicability can significantly reduce efficiency and inflate computational costs.

To address this challenge, we propose a hybrid interval method that strategically combines extrinsic and intrinsic approaches. Within this framework, different subsets of interval parameters are assigned to the most suitable solver: optimization-based methods are used to handle nonlinear parameterizations, while more efficient intrinsic or guaranteed algorithms are employed for parameters where they are applicable. This selective integration leverages the complementary strengths of both classes of methods, aiming to improve computational efficiency without significantly sacrificing accuracy or reliability of the solution bounds.

The remainder of this paper is organized as follows. Section 2 provides a concise review of existing methods for interval structural analysis and highlights their main properties. Section 3 introduces the formulation of the proposed hybrid interval method, detailing how the partitioning of parameters between solvers is achieved. Section 4 presents numerical studies on truss structures with up to 434 independent interval parameters, demonstrating the effectiveness and scalability of the hybrid approach. Finally, Section 5 summarizes the key findings and discusses the implications for practical applications

2. Review of Previous Work

Structural analysis using interval variables has been explored by several researchers to account for uncertainty in their analyses (Köylüoglu, Cakmak et al. 1995; Muhanna and Mullen 1995; Nakagiri and Yoshikawa 1996; Rao and Berke 1997; Qiu and Elishakoff 1998; Rao and Chen 1998; Neumaier and Pownuk 2007, Moens, D. Vandepitte, D., 2002 among others). These studies primarily focused on achieving meaningful results without the excessively large and unrealistic solution bounds that can arise from the naive application of interval arithmetic.

The earliest known work on interval linear systems is credited to Oettli and Prager (1964). Since then, a wide range of methods have been developed (Neumaier 1990), reflecting different strategies for dealing with the fundamental challenge of enclosing all possible solutions within bounded uncertainty. In structural mechanics, at least five main classes of solution techniques have emerged: optimization, Monte Carlo simulation, fixed-point enclosures, perturbation methods, and endpoint/sensitivity methods. Each of these approaches offers distinct strengths, limitations, and areas of application.

Optimization-based methods solve interval problems by framing them as constrained optimization problems, where the bounds of response quantities are the objective functions, and the interval parameters are the independent variables with inequality constraints representing the interval bounds (Rao and Chen 1998). Their key advantage lies in their extrinsic nature, which enables the direct use of existing non-interval structural analysis software. Moreover, optimization approaches can be extended to nonlinear systems, provided the underlying analysis program supports such formulations (Möller and Beer 2004). However, optimization has notable drawbacks: each response quantity typically requires a separate optimization run, the problems are NP-hard (Rohn and Kreinovich 1995), and solutions can be computationally expensive. In addition, depending on numerical tolerances, optimization may yield inner bounds, enclosures smaller than the true solution set, thus losing the guarantee of correctness. A recent direction has been the integration of Bayesian optimization techniques (Dang, Wei et al. 2022), which improve efficiency but retain the fundamental limitations of optimization-based strategies.

Monte Carlo methods, in contrast, adapt probabilistic simulation to the interval setting by interpreting interval parameters as random variables—often with uniform distributions. While conceptually straightforward and extrinsic in nature, Monte Carlo does not produce guaranteed enclosures and may require very large numbers of realizations even for low-dimensional problems. Initially, Monte Carlo was used primarily for benchmarking (Köylüoglu, Cakmak et al. 1995; Muhanna, Mullen et al. 2005). More recently, advances such as Quasi-Monte Carlo sampling with Cauchy distributions (Callens, Faes et al. 2021) have significantly reduced computational costs, making Monte Carlo competitive with vertex methods in certain applications.

Perturbation methods approximate interval equations by expressing the interval stiffness matrix and load vector as the sum of midpoint values and perturbations. A truncated Taylor expansion, typically first- or second order, is used to construct approximate equilibrium equations. These methods (Qiu and Elishakoff 1998; McWilliam 2000; Qiu and Elishakoff 2001; Chen, Lian et al. 2002) are efficient for small perturbations but cannot guarantee enclosures. Their results are inner estimates of the solution set, which can underestimate the true uncertainty in structural responses.

Sensitivity or endpoint methods assume that extreme responses occur at the endpoints of each interval parameter. Naive implementation requires 2^N evaluations (with N interval parameters), which quickly becomes intractable. However, Rohn (1989) introduced an efficient procedure requiring only 2N evaluations, significantly improving feasibility. Subsequent applications in structural analysis (Popova, Datcheva et al. 2002; Pownuk 2004; Popova, Iankov et al. 2006) demonstrated their utility, and further computational refinements have been reported (Sofi and Romeo 2016; Popova 2020). Endpoint methods generally yield inner solutions, but in special cases, such as when an interval parameter multiplies a rank-one matrix and is parameterized linearly, they can produce exact solutions (Popova 2017; Popova 2020; Popova 2021).

Finally, fixed-point methods represent a fundamentally different, intrinsic approach. Based on fixed-point theorems, these methods have been shown to provide guaranteed enclosures of the solution set (Gay 1982;

Jansson 1991; Dessombz, Thouverez et al. 2001; Neumaier and Pownuk 2007). Moreover, they can also be adapted to produce inner approximations, thereby quantifying the degree of overestimation (Xiao 2015). A major advantage is that fixed-point approaches provide the complete solution set for all variables simultaneously, rather than requiring separate computations for each quantity of interest.

A concise comparison of the properties, strengths, and limitations of these five classes of methods is provided in Table I.

Table I. Summary of properties of interval equation solvers.

Property	Fixed-Point	Monte Carlo	Optimization	Perturbation	Sensitivity
Guaranteed enclosure	✓	✗	✗	✗	✓*
Extrinsic	✗	✓	✓	✗	✓
Nonlinear problems	✓**	✓	✓	✗	✗
One response at a time	✗	✗	✓	✗	✓

** Fixed-point: limited nonlinear (e.g., truss nonlinear materials).
* Sensitivity: exact only in problems that possess the endpoint property.

3. General Description of the Hybrid Method

The primary objective of this work is to enhance the efficiency without significant degradation of the accuracy of solving interval structural analysis problems. This is accomplished by dividing the interval parameters into two subsets. One subset is evaluated using an extrinsic method based on optimization, while the other subset is evaluated using an intrinsic method embedded directly within the optimization objective function. The goal is to assign each parameter to the solution strategy that best suits its characteristics.

For example, when an interval parameter satisfies the endpoint property, an intrinsic sensitivity method can provide exact results more efficiently than an optimization-based approach. Conversely, interval parameters with nonlinear dependencies or no endpoint property are better suited for optimization-based evaluation.

The hybrid method under investigation, therefore, integrates two levels of solution strategies:

- Upper-level optimization (extrinsic): Interval parameters without intrinsic structure are solved using optimization algorithms such as Particle Swarm Optimization (PS), gradient-based solvers (e.g., fmincon function in MATLAB), Genetic Algorithms (GA), or Bayesian Optimization. These algorithms explore the admissible interval space to determine extremal values of the response.
- Embedded intrinsic analysis: Within the optimization function, interval parameters that lend themselves to fixed-point iteration (FP) or sensitivity methods (SM) are solved directly, exploiting their structure for improved accuracy and reduced computational cost.

By combining these approaches, the hybrid framework ensures that each subset of interval parameters is handled by the most appropriate method, balancing efficiency and reliability of enclosures.

The general implementation of the hybrid method proceeds as outlined by the following steps:

1. Partition Interval Parameters: Separate all interval parameters into two distinct groups, intrinsic and extrinsic, ensuring that each parameter belongs exclusively to one group.
2. Formulate the Objective Function: Construct the objective function for a chosen response variable using the structural model. In the structural model, intrinsic parameters are treated as intervals, while extrinsic parameters are given scalar values within their prescribed ranges.
3. Evaluate the Objective Function: Each evaluation produces an interval response. When optimization seeks a minimum (maximum), the lower (upper) bound of this interval is used as the objective value.
4. Update Extrinsic Variables: The optimization algorithm adjusts the scalar values of the extrinsic parameters, subject to interval constraints, to minimize or maximize the objective function.
5. Iterate Optimization: Steps 3 and 4 are repeated until the algorithm satisfies its termination criteria.
6. Compute Bounds for Each Response: Repeat steps 2–5 to determine both the lower and upper bounds for every response variable of interest.

In traditional optimization-based approaches to interval problems, structural responses are evaluated using a deterministic analysis method. In this work, we replace the deterministic evaluation with interval analysis for a selected intrinsic subset of interval parameters. The allocation of each parameter to either the extrinsic (optimization-based) or intrinsic (interval-based) subset is determined by the user, depending on whether the parameter can be effectively represented through an intrinsic interval solver.

For the intrinsic interval solver, we employ sensitivity methods and fixed-point methods. In particular, for sensitivity analysis, we adopt Popova's algorithm, where a single iteration of the fixed-point expansion identifies the appropriate endpoint (SM). For the fixed-point (FP) method, we follow the formulation outlined in Xiao (2015). Both intrinsic methods, however, are constrained by the way interval quantities parameterize the problem. Interval parameters that cannot be accommodated by the FP or SM intrinsic solvers are instead handled through optimization.

This hybrid allocation reduces the dimensionality of the optimization problem, thereby improving computational efficiency while maintaining reliable interval enclosures.

4. Specific Formulation for a Hybrid Gradient-–Sensitivity Method

Without loss of generality, the optimization method is formulated as a constrained minimization problem. A wide range of computational tools can address such problems, including MATLAB (2026), PROFIL/BIAS Knüppel, O. (1994), and Bayesian optimization (Dang 2022). The optimization process requires identifying the independent extrinsic parameters, defining their lower and upper bounds, and specifying an objective function to evaluate the structural response. In this study, we employ MATLAB's gradient-based solver (fmincon), though other optimization algorithms, such as Particle Swarm Optimization or Genetic Algorithms could also be used.

The fmincon algorithm explores candidate scalar values for the extrinsic parameters within their prescribed bounds and passes them to the objective function. Within this function, a sensitivity-based interval solver evaluates the lower or upper bound of the structural response. This hybrid evaluation couples the extrinsic optimization with intrinsic methods, enabling efficient handling of mixed parameter types. Specifically, interval parameters that can be addressed using sensitivity methods (SM) or fixed-point methods

(FP) are treated intrinsically, while parameters unsuitable for intrinsic analysis are assigned to the extrinsic set.

5. Results

In the following, solutions to three representative problem scenarios are presented. All of the intrinsic solutions are calculated using the method presented by Rama Rao et.al. (2011) and interval calculations are performed using INTLAB (Rump 1999). The first set is a simple truss structure for which an analytical solution is readily calculated. The hybrid formulation treats horizontal bars using a call to a non-linear incremental plasticity model. To preserve the computable analytic solution, the yield stress is set to eliminate any plastic behavior. The quality of the interval enclosures is studied. The second example set considers a truss structure subjected to applied loads with interval-valued angles and magnitudes. This problem was selected because it is well known that endpoint-based methods, such as sensitivity approaches, fail to produce correct interval enclosures when uncertainty appears in angular parameters. In addition, nonlinear parameterizations of loading terms are not adequately addressed by fixed-point methods. Finally, we solve a truss with the extrinsic parameters describing a nonlinear constitutive law.

The first example problem is a simple five-bar two-dimensional truss shown in Figure 1. The horizontal bars are 4 m in length, the vertical bars are 3 m, and the diagonal element has a length of 5 m. The bottom-left node is pinned, and the top-left node is constrained in the horizontal direction. All bars have a nominal Young's modulus of 2.0×10^8 kPa and a cross-sectional area of 0.001 m^2. The downward load at node 2 is 10 kN.

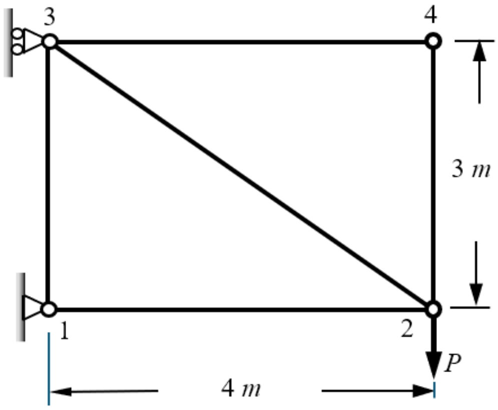


Figure 1: Simple truss structure

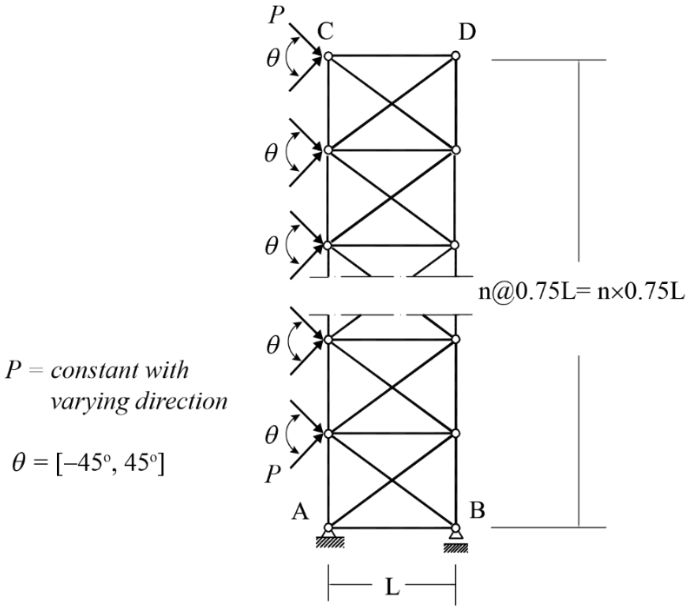
All bars have an interval uncertainty of ± 0.04 for the modulus and ± 0.01 for the area. The load uncertainty is ± 0.05 In the hybrid calculations, bars 1-2 and 3-4 were treated using an incremental plasticity law with the yield stress large enough to prevent the elements from becoming plastic. The pure optimization (fmincon) and the hybrid fmincon-sensitivity results for the vertical deflection at node 2 are compared in Table II.

Table II: Comparison of interval enclosure of optimization and hybrid solutions

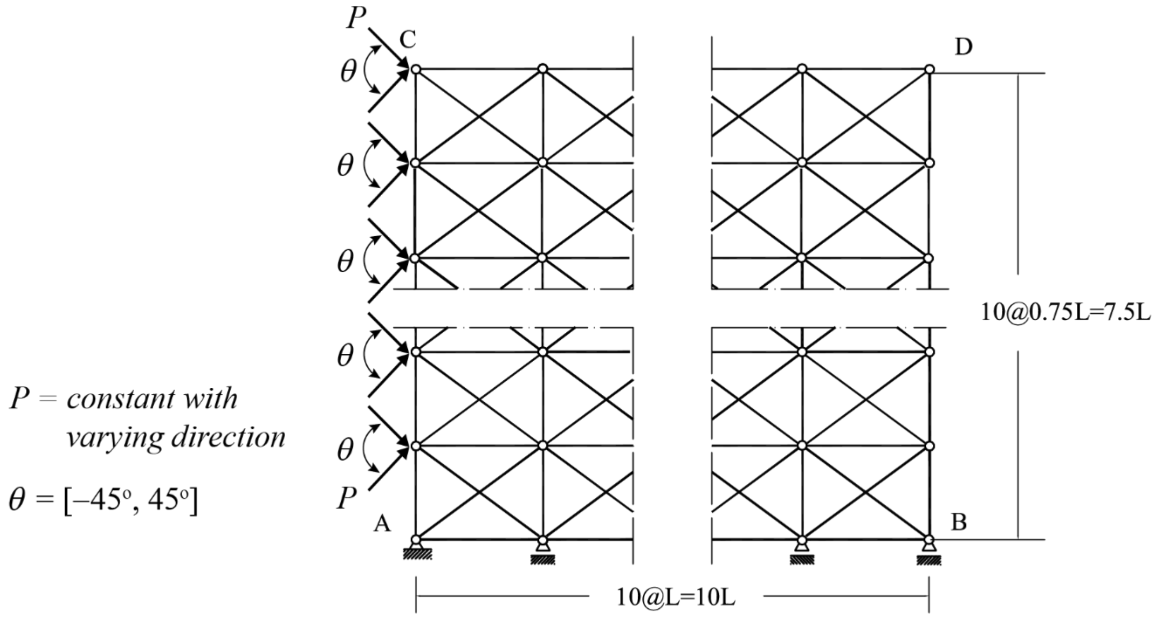
all units in m	fmincon	hybrid	exact
inf Uy_2	-0.00132305	-0.001325363	-0.001325758
sup Uy_2	-0.001087973	-0.001085473	-0.001085301
interval width	0.000235077	0.00023989	0.000240457
inf Uy_2 from exact	2.708E-06	3.95E-07	0
sup Uy_2 from exact	2.672E-06	1.72E-07	0

Neither the hybrid nor the optimization solutions enclose the exact solution. However, the underestimation of the interval width is extremely small in both calculations. Increasing the number of interval variables treated by the optimization algorithm increases the underestimation of the interval width. In this example, using the sensitivity method (which provides a guaranteed enclosure when used alone) to model some of the variables decreases the underestimation introduced by the optimization method

The second set of problems used to investigate the behavior of the proposed hybrid approach consists of a series of truss structures that have been widely employed to evaluate interval



(a)



(b)

Figure 2. a) a single bay and b) a multiple bay cross-braced truss structure.

structural solvers Muhanna, R. L. (2004). These benchmark problems are shown in Figure 2. The traditional problem configuration is modified by introducing a nonlinear parametric loading, in which the direction of the applied load is uncertain and varies within an interval of ±45° from the horizontal.

This type of interval uncertainty in the loading angle is well-suited for treatment via optimization-based methods. Accordingly, the interval parameter associated with the loading direction is treated extrinsically, while interval uncertainties associated with element stiffness and load magnitude are handled intrinsically using either fixed-point or sensitivity-based methods.

The primary structure considered is a one-bay, 20-floor cantilever truss with a bay length of $L = 1$m and a story height of $0.75L$. The model consists of 42 nodes and 101 elements. The truss is subjected to 20 horizontal concentrated forces applied at the left-hand-side nodes, each acting at an uncertain angle between $-45°$ and +45° relative to the horizontal. The midpoint magnitude of each load is $P = 10$kN. The midpoint values of Young's modulus and cross-sectional area are taken as $E = 200$GPa and $A = 1.0 \times 10^{-3}$ m^2, respectively, and are assumed to be independent for each element.

To study the effect of problem size, reduced versions of the structure with 1, 5, and 10 floors are also analyzed. In addition, a larger ten-bay, ten-story truss (Figure 2b) is included to represent problems with a substantially increased number of interval parameters. This model has bay and floor dimensions of L and $0.75L$, respectively, and consists of 121 nodes and 420 elements.

The number of interval parameters for each model is summarized in Table III. For example, the 1×1 truss includes six interval parameters associated with element stiffness, one interval parameter for load magnitude, and one for load direction, for a total of eight interval parameters. Interval parameter values are presented in Table IV. All five models are analyzed assuming independent element properties, with 6% uncertainty in Young's modulus, 10% uncertainty in each load magnitude, and a common uncertain load angle varying between −45° and +45°. For all cases, the horizontal displacement of the upper right-hand node is selected as the response quantity of interest.

Table III: Measure of problem size for problem set two.

Description	Number of elements	Number of nodes	Number of interval parameters
1x1 truss structure	6	4	8
1x5 truss	26	12	32
1x10	51	22	62
1x20	101	42	122
10x10	420	101	431

Table IV: Values of Interval parameters (LB is lower bound, UB is upper bound).

Parameter	LB	UB
Young's modulus (kPa)	1.96×10^8	2.04×10^8
Force magnitude (kN)	9.5	10.5
Force angle (degrees)	−45	+45

A summary of the numerical results is presented in Table V. The optimization routine used for the extrinsic calculations is MATLAB's fmincon, with default parameters. No attempt was made to improve computational performance by tuning the optimizer settings. The methods reported in the table are denoted as follows:

- fmin: pure optimization applied to all interval parameters.
- fmin–fp: hybrid approach using fmin for the extrinsic set and a fixed-point method for the intrinsic set.
- fmin–sen: hybrid approach using fmin for the extrinsic set and a sensitivity-based method for the intrinsic interval parameters.

Table V: Comparison of optimization and hybrid approaches for truss structures with uncertain parameters.

Interval parameters	Method	Lower bound	Upper bound	clock time (sec)	Processing time (sec)	Number of function calls	Number of floors	Number of bays	Ratio of function calls	Ratio of processing time	Ratio of interval width
8	fmin	3.34E-05	6.31E-05	0.221405	0.3125	326	1	1	1	1.0	1.00
32	fmin	0.002382	0.00553	1.344753	2.75	7658	5	1	1	1.0	1.00
62	fmin	0.034261	0.068537	11.03258	12.03125	41051	10	1	1	1.0	1.00
122	fmin	0.543175	0.989051	454.5347	463.7813	951339	20	1	1	1.0	1.00
431	fmin	4.79E-05	0.000107	33.92403	178.3438	18144	10	10	1	1.0	1.00

1-7	fmin-fp	3.15E-05	6.52E-05	0.123817	0.234375	45	1	1	7	1.3	0.88
1-31	fmin-fp	0.002254	0.00554	0.357928	1.015625	44	5	1	174	2.7	0.96
1-61	fmin-fp	0.033351	0.068554	0.593787	1.109375	44	10	1	933	10.8	0.97
1-121	fmin-fp	0.534761	0.989074	0.850719	1.90625	35	20	1	27181	243.3	0.98
1-430	fmin-fp	4.19E-05	0.000114	4.824301	12.6875	47	10	10	386	14.1	0.81
1-7	fmin-sen	3.19E-05	6.57E-05	0.028332	0.03125	65	1	1	5	10.0	0.88
1-31	fmin-sen	0.002359	0.005541	0.063423	0.140625	85	5	1	90	19.6	0.99
1-61	fmin-sen	0.034237	0.068556	0.035624	0.1875	46	10	1	892	64.2	1.00
1-121	fmin-sen	0.543067	0.989095	0.062328	0.40625	73	20	1	13032	1141.6	1.00
1-430	fmin-sen	4.44E-05	0.000115	0.799103	8.140625	88	10	10	206	21.9	0.84

All the ratios given in the tables are the hybrid methods/optimization methods. In all cases, the hybrid approaches required less computational effort than the pure optimization method. The ratio of processing time for the pure optimization method to that of the hybrid methods ranged from approximately 1.3 to 1142, indicating substantial computational savings. In general, the computational advantage of the hybrid approach increased with the number of intrinsic interval parameters. An exception was observed in the pure optimization results, where the one-bay, 20-story truss required approximately four times the computation time of the ten-bay, ten-story truss; the reason for this behavior could not be identified. Sensitivity-based hybrid methods consistently required less computational effort than their fixed-point counterparts.

The interval widths of the hybrid calculations are wider than those of the pure optimization calculations. While we have not calculated exact solutions for this problem, we speculate that neither the optimization nor the hybrid solutions provide guaranteed bounds and that the hybrid solutions are closer to the exact solution.

To investigate the influence of the magnitude of parameter uncertainty, the same set of problems was recalculated using a 20% uncertainty for all interval variables, except for the load direction. The corresponding results are presented in Table VI.

Table VI: Hybrid results with 20% uncertainty.

Number of floors	Number of bays	Interval parameters	Method	Lower bound	Upper bound	clock time (sec)	Processing time (sec)	Number of function calls	Ratio of function calls	Ratio of processing time	Ratio of interval width
1	1	8	fmin	3.22E-05	6.58E-05	0.0606076	0.109375	387	1	1.0	1.0
5	1	32	fmin	0.002271	0.005951836	1.6487626	1.71875	9683	1	1.0	1.0
10	1	62	fmin	0.030407	0.077374204	14.489049	17.6718	53072	1	1.0	1.0
20	1	122	fmin	0.48195	1.116704863	643.5102	655.515	1326891	1	1.0	1.0
10	10	431	fmin	4.62E-05	0.000113	51.454384	263	27648	1	1.0	1.00
1	1	1-7	fmin-fp	2.56E-05	7.39E-05	0.1490789	0.39062	50	8	0.3	0.70

5	1	1-31	fmin-fp	0.001804	0.006251902	0.353652	0.76562	47	206	2.2	0.83
10	1	1-61	fmin-fp	0.027487	0.0773436	1.0318756	1.89062	44	1206	9.3	0.94
20	1	1-121	fmin-fp	0.447599	1.115793432	0.9589624	1.84375	39	34023	355.5	0.95
10	10	1-430	fmin-fp	3.25E-05	0.000129467	4.9069113	11.2031	47	588	23.5	0.69
1	1	1-7	fmin-sen	2.71E-05	7.56E-05	0.0333906	0.03125	67	6	3.5	0.69
5	1	1-31	fmin-sen	0.00209	0.006257308	0.0385201	0.04687	89	109	36.7	0.88
10	1	1-61	fmin-sen	0.030368	0.077407429	0.0273507	0.15625	48	1106	113.1	1.00
20	1	1-121	fmin-sen	0.481737	1.116789039	0.0532632	0.29687	46	28845	2208.	1.00
10	10	1-430	fmin-sen	3.86E-05	0.000130345	0.6508958	6.76562	74	374	38.9	0.73

*For hybrid models, the number of intervals is an extrinsic-intrinsic pair. **Method is fmin – Newton-based method, sen- Sensitivity, fp – Fixed point method

When using pure optimization, the number of function evaluations increased with larger interval parameter widths, and for all cases except the 1×1 and 5×1 trusses, both wall-clock and processor times increased accordingly. In contrast, the hybrid methods exhibited relatively stable computational effort with respect to increased parameter uncertainty. The number of function calls and both timing measures remained largely unchanged, demonstrating the robustness of the hybrid approach with respect to uncertainty magnitude. In all results, the interval width of the hybrid calculations is larger, consistent with the previous examples.

The next group of calculations considers the case in which the direction of each applied load is treated as an independent interval parameter. This significantly increases the number of extrinsic interval variables relative to the previous cases. The results are summarized in Table VII.

Table VII: Solution behavior with independent loading directions.

Number of floors	Number of bays	Interval parameters	Method	Lower bound	Upper bound	clock time (sec)	Processing time (sec)	Number of function calls	Ratio of function calls	Ratio of processing	Ratio of interval width
1	1	8	fmin	3.34E-05	6.31E-05	0.133373	0.25	326	1	1.0	1.00
5	1	36	fmin	0.002384	0.005567	1.712985	2.40625	10853	1	1.0	1.00
10	1	71	fmin	0.034264	0.068821	15.20872	18.8125	60246	1	1.0	1.00
20	1	141	fmin	0.543149	0.990547	354.8577	465.5313	736875	1	1.0	1.00
10	10	440	fmin	4.75E-05	0.00011	193.0471	955.2813	97466	1	1.0	1.00
1	1	1-7	fmin-fp	3.19E-05	6.57E-05	0.137894	0.125	65	5	2.0	0.88
5	1	5-31	fmin-fp	0.002359	0.005579	0.174381	0.390625	649	17	6.2	0.99
10	1	10-61	fmin-fp	0.034237	0.06884	0.249899	2.09375	1527	39	9.0	1.00
20	1	20-121	fmin-fp	0.543067	0.990598	1.171556	9.421875	4387	168	49.4	1.00

10	10	10-430	fmin-fp	9.04E-05	0.000114	13.70857	145.3281	1779	55	6.6	2.61
1	1	1-7	fmin-sen	3.14E-05	6.52E-05	0.132542	0.25	45	7	1.0	0.88
5	1	5-31	fmin-sen	0.002251	0.005578	2.992782	5.515625	443	24	0.4	0.96
10	1	10-61	fmin-sen	0.03333	0.068838	17.43458	28.42188	1454	41	0.7	0.97
20	1	20-121	fmin-sen	0.534599	0.990576	79.33231	113.4688	3467	213	4.1	0.98
10	10	10-430	fmin-sen	4.28E-05	0.000114	107.7495	187.2031	1209	81	5.1	0.87

In all but three cases, the hybrid methods outperformed the corresponding pure optimization approaches in terms of computational efficiency. The exceptions occurred for the fmin–sensitivity method applied to the 1×1, 5×1, and 10×1 structures, where processor times were comparable to or slightly shorter for pure optimization. Nevertheless, wall-clock times and the number of function evaluations remained favorable for the hybrid methods.

The third example considers a hybrid formulation in which optimization-based methods are employed to address nonlinear elements, while the remaining linear elements are treated using an intrinsic sensitivity-based formulation. Elements 7 and 9 are considered to be nonlinear. The geometry and boundary conditions of the two-dimensional truss are shown in Figure 3. The joints are numbered from 1 to 8, and the elements are labeled with an underscore from 1 to 15.

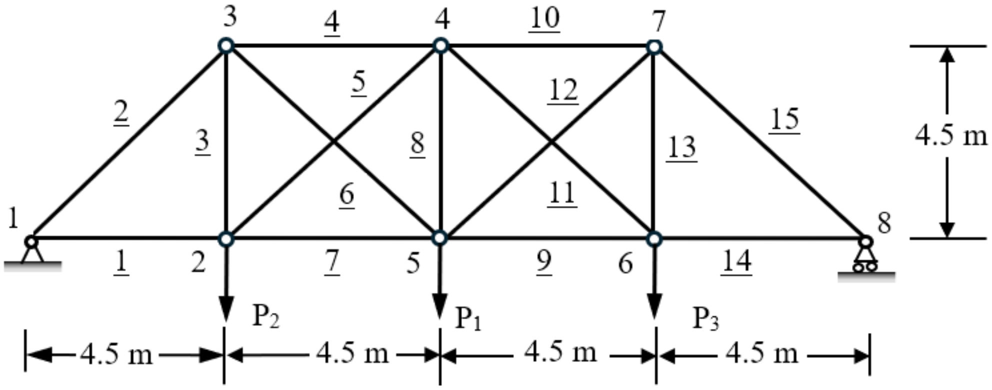


Figure 3. Fifteen-bar truss

Point loads P_1, P_2, and P_3, are applied at joints 5, 2, and 6, respectively. The two central bottom elements are modeled using an incremental plasticity formulation with uncertain constitutive parameters listed in Table VIII. The midpoint values of the applied loads are $P_1 = 200\,\text{kN}$ and $P_2 = P_3 = 100\,\text{kN}$, each with an uncertainty of ±25%.

Table VIII: Incremental plasticity model parameters

Parameter	Lower limit	Upper limit
Elastic slope (kPa)	190×10^6	210×10^6
Yield stress (kPa)	300×10^3	400×10^3
Plastic slope/Elastic slope	0.015	0.025

Bars 1– 4, 10, and 13–15 have a cross-sectional area of $A = 1.2 \times 10^{-3}\ \mathrm{m}^2$, while all remaining bars have a smaller area of $A = 0.6 \times 10^{-3}\ \mathrm{m}^2$. All cross-sectional areas are assumed to have ±1% interval uncertainty. All bars are made of steel and are assigned an interval Young's moduli with midpoint value $E = 200\mathrm{GPa}$ and ±5% uncertainty bounds. Uncertainties in material properties and loads are assumed to be independent.

Three response quantities are evaluated:

1. the vertical deflection at node 5,
2. the lateral deflection at node 8, and
3. the axial force in element 10.

Table IX reports the lower and upper bounds of these response quantities for the different solution algorithms. The reported computational times correspond to the authors' implementations executed on the computer system described previously. The problem includes a total of 37 interval parameters, consisting of 13 Young's moduli for elastic elements, 15 cross-sectional areas, 3 nodal loads, and 6 constitutive parameters associated with the two elements exhibiting incremental plasticity.

The pure optimization results are compared with a hybrid formulation in which the optimization method is applied only to the six interval parameters associated with the potentially nonlinear behavior of elements 7 and 9.

Table IX: Non-linear truss results

Interval parameters	Method	Variable	Lower bound	Upper bound	clock time (sec)	Processing time (sec)	Number of function calls	Ratio of function calls	Ratio of processing time	Ratio of interval width
39	fmin	U5y	-0.09693	-0.0295	69.05129	320.4688	7707	1	1.0	1.000000
6-33	fmin-sen	U5y	-0.10105	-0.02914	4.130456	4.375	175	44.04	73.3	0.937663
39	fmin	U8x	0.016827	0.089442	67.42967	314.0625	8153	1	1.0	1.000000
6-33	fmin-sen	U8x	0.01632	0.092603	4.458789	4.203125	196	41.597	74.7	0.951918
39	fmin	F10	-436.824	-206.471	19.39599	106.0938	1641	1	1.0	1.000000
6-33	fmin-sen	F10	-445.164	-205.457	5.590322	5.734375	222	7.3919	18.5	0.960978

From a computational perspective, the optimization-only approach required between 18 and 75 times more processing time, depending on the response variable. In addition, the number of function evaluations for the optimization-only method was between 7 and 44 times more than that required by the hybrid approach. Also, as seen in previous examples, the widths of the hybrid results are wider than those of the pure optimization.

6. Conclusions

The simultaneous (hybrid) use of extrinsic and intrinsic methods, where each interval variable is assigned to one of the methods, is presented. Both uncertain loading directions, as well as material nonlinear behavior, are presented. In most scenarios studied, the use of a hybrid approach is computationally more efficient. While both optimization and hybrid methods do not provide a guaranteed enclosure, they both provide similar results, with the hybrid methods providing slightly larger interval widths compared to the optimization method.

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