Hybrid Reliability Analysis with Uncertainty in Correlation Coefficients and First Two Statistical Moments

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Abstract. Traditional reliability analysis aims to evaluate the failure probability of structures under the presence of aleatory uncertainties. However, in real-world applications, limited data often introduces epistemic uncertainties, requiring the use of hybrid reliability analysis. In this study, to address the epistemic uncertainty caused by insufficient data, both the statistical moments and the correlation coefficients of the random variables are uncertain, and are represented using interval variables. With the aid of the simplified third-moment normal transformation model, the values of uncertain moments and correlation coefficients of each random variables that lead to the bounds of failure probability are determined. The bounds of the failure probability are then evaluated through two traditional reliability analyses. The proposed method is demonstrated through two examples.

Keywords: Reliability analysis, Epistemic uncertainty, Correlation coefficient, First two moments.

1. Introduction

Structural reliability analysis aims to evaluate the failure probability of a structure in the presence of uncertainty. These uncertainties are usually quantified by the probabilistic information of random variables; in traditional reliability analysis, the statistical moments of random variables and the correlation coefficients among random variables are typically treated as exact values. It is reasonable when sufficient observations are available, but it is often difficult to justify in practical engineering. For some complex structures in engineering, usually only limited data can be obtained.

This situation involves both aleatory uncertainty (inherent variability) and epistemic uncertainty (e.g., limited samples or incomplete knowledge) (Der Kiureghian, 2008; Beer et al., 2013). Hybrid reliability analysis provides a framework for considering these uncertainties together. To handle epistemic uncertainty, key information such as statistical moments or correlation coefficients are treated as intervals (Beer et al., 2013; Nannapaneni and Mahadevan, 2016; Wang et al., 2024a). Consequently, the reliability result is expressed as an interval of failure probability or reliability index.

Several studies have investigated reliability analysis methods that consider uncertain statistical moments as intervals (i.e., parametric probability boxes) (Faes et al., 2021; Wang et al., 2024b; Wang et al., 2026). Similar to the uncertain statistical moments, there are also methods considering uncertain correlation coefficients as intervals (Zaman et al., 2013; Nannapaneni and Mahadevan, 2016). However, when the probability information of random variables is incomplete, the uncertainties of both statistical moments and correlation coefficients should be considered simultaneously. Currently, there is only limited research addressing this, such as methods using auxiliary variables and a Bayesian probabilistic framework to include distribution parameter uncertainty and correlation uncertainty in MCS- and FORM-based reliability analysis (Nannapaneni and Mahadevan, 2016). The computational cost of such methods increases significantly as the number of uncertain parameters increases.

Building upon the foundation established by TIPP method (Wang et al., 2024b) and the method proposed in (Wang et al., 2024a), this paper proposes a hybrid reliability analysis method for random variables with interval first two moments and interval correlation coefficients. The rest of this paper is as follows. Section 2 gives the problem statement. Then, section 3 proposes the method considering uncertain statistical moments and uncertain correlation coefficients, including the detailed theoretical derivation, as well as summarizing the straightforward procedure. Section 4 demonstrates the method through two examples. Finally, Section 5 summarizes the key findings, discusses potential applications in engineering practice and outlines the future works.

2. Problem Statement

Consider a performance function

$$G(\boldsymbol{X}) = G(X_1, X_2, \ldots, X_n), \tag{1}$$

where $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)^\mathsf{T}$ is a vector of correlated random variables. The failure domain is defined as $G(\boldsymbol{X}) \leq 0$, and the safe domain is defined as $G(\boldsymbol{X}) > 0$. The failure probability is

$$P_f = \int_{\Omega_{\boldsymbol{X}}} I_{\boldsymbol{X}}(\boldsymbol{x}) f_{\boldsymbol{X}}(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}, \tag{2}$$

where $f_{\boldsymbol{X}}(\boldsymbol{x})$ is the joint probability density function of $\boldsymbol{X}$, and $I_{\boldsymbol{X}}(\boldsymbol{x})$ is an indicator function written as

$$I_{\boldsymbol{X}}(\boldsymbol{x}) = \begin{cases} 1, & G(\boldsymbol{x}) \leq 0, \\ 0, & G(\boldsymbol{x}) > 0. \end{cases} \tag{3}$$

In this study, the first two moments and selected correlation coefficients are not treated as fixed values. The mean and standard deviation of X_i are denoted by θ_{i1} and θ_{i2}, respectively, and are represented by intervals

$$\theta_{i1} \in [\underline{\theta}_{i1}, \overline{\theta}_{i1}], \qquad \theta_{i2} \in [\underline{\theta}_{i2}, \overline{\theta}_{i2}]. \tag{4}$$

Most random variables follow two-parameter distributions, and their skewness α_{3i} is a known value. The uncertain correlation coefficient between X_i and X_j is denoted by

$$\lambda_{ij} \in [\underline{\lambda}_{ij}, \overline{\lambda}_{ij}]. \tag{5}$$

All uncertain moments and correlation coefficients are collected in the interval vectors $\boldsymbol{\Theta}^I$ and $\boldsymbol{\Lambda}^I$.

Because of these interval parameters, the failure probability is no longer a single value. It becomes a function of $\boldsymbol{\Theta}$ and $\boldsymbol{\Lambda}$:

$$P_f(\boldsymbol{\Theta}, \boldsymbol{\Lambda}) = \int_{\Omega_{\boldsymbol{X}}} I_{\boldsymbol{X}}(\boldsymbol{x}) f_{\boldsymbol{X}}(\boldsymbol{x} \mid \boldsymbol{\Theta}, \boldsymbol{\Lambda}) \mathrm{d}\boldsymbol{x}. \tag{6}$$

The target of hybrid reliability analysis is to estimate the interval

$$P_f^I = [\underline{P}_f, \overline{P}_f], \qquad \beta^I = [\underline{\beta}, \overline{\beta}], \tag{7}$$

3. Proposed Method

3.1. Bounds of Failure Probability

Based on Eq. (6), the uncertainty in $\boldsymbol{\Theta}$ and $\boldsymbol{\Lambda}$ is propagated through $f_{\boldsymbol{X}}(\boldsymbol{x} \mid \boldsymbol{\Theta}, \boldsymbol{\Lambda})$ to $P_f(\boldsymbol{\Theta}, \boldsymbol{\Lambda})$. To facilitate the derivation, $\boldsymbol{X}$ is mapped into a vector of independent standard normal random variables $\boldsymbol{U}$ based on the statistical moments and correlation coefficients, and then $P_f(\boldsymbol{\Theta}, \boldsymbol{\Lambda})$ is redefined in Gaussian space as follows:

$$P_f(\boldsymbol{\Theta}, \boldsymbol{\Lambda}) = \int_{\Omega_{\boldsymbol{U}}} I_{\boldsymbol{U}}(\boldsymbol{u} | \boldsymbol{\Theta}, \boldsymbol{\Lambda}) \phi_{\boldsymbol{U}}(\boldsymbol{u}) \mathrm{d}\boldsymbol{u}, \tag{8}$$

where $\Omega_{\boldsymbol{U}} \in \mathbb{R}^n$ denotes the domain of $\boldsymbol{U}$; $\phi_{\boldsymbol{U}}(\boldsymbol{u})$ is the JPDF of $\boldsymbol{U}$, which follows a standard Gaussian distribution in n dimensions; and $I_{\boldsymbol{U}}(\boldsymbol{u} | \boldsymbol{\Theta}, \boldsymbol{\Lambda})$ is the indicator function defined in Gaussian space, which is defined as follows:

$$I_{\boldsymbol{U}}(\boldsymbol{u} | \boldsymbol{\Theta}, \boldsymbol{\Lambda}) = I_{\boldsymbol{X}}[S\left(\boldsymbol{u} | \boldsymbol{\Theta}, \boldsymbol{\Lambda}\right)], \tag{9}$$

where $S\left(\boldsymbol{u} | \boldsymbol{\Theta}, \boldsymbol{\Lambda}\right)$ is the inverse normal transformation function defined based on the first three moments and correlation coefficients of $\boldsymbol{X}$.

Based on Eq. (8), P_f^I can be expressed as:

$$\underline{P}_f = \int_{\Omega_{\boldsymbol{U}}} I_{\boldsymbol{U}}(\boldsymbol{u} | \boldsymbol{\Theta}_{\text{low}}, \boldsymbol{\Lambda}_{\text{low}}) \phi_{\boldsymbol{U}}(\boldsymbol{u}) \mathrm{d}\boldsymbol{u}, \tag{10}$$

$$\overline{P}_f = \int_{\Omega_{\boldsymbol{U}}} I_{\boldsymbol{U}}(\boldsymbol{u} | \boldsymbol{\Theta}_{\text{up}}, \boldsymbol{\Lambda}_{\text{up}}) \phi_{\boldsymbol{U}}(\boldsymbol{u}) \mathrm{d}\boldsymbol{u}. \tag{11}$$

where $\bar{\cdot}$ and $\underline{\cdot}$ are the upper and lower bounds, respectively; and $(\boldsymbol{\Theta}_{\text{low}}, \boldsymbol{\Lambda}_{\text{low}})$ and $(\boldsymbol{\Theta}_{\text{up}}, \boldsymbol{\Lambda}_{\text{up}})$ are the realizations of moments and correlation coefficients that lead to the lower and upper bounds of P_f, respectively. Then, the value of β^I can be obtained based on P_f^I as follows:

$$\beta^I = [\underline{\beta}, \overline{\beta}] = \left[-\Phi^{-1}\left(\overline{P}_f\right), -\Phi^{-1}\left(\underline{P}_f\right)\right]. \tag{12}$$

As shown in Eqs. (10)–(12), the key task in determining β^I and P_f^I lies in finding $(\boldsymbol{\Theta}_{\text{low}}, \boldsymbol{\Lambda}_{\text{low}})$ and $(\boldsymbol{\Theta}_{\text{up}}, \boldsymbol{\Lambda}_{\text{up}})$, which will be discussed in detail in the next section.

3.2. Monotonicity Analysis

The simplified third-moment normal transformation is used to map random variables into Gaussian space (Qin et al., 2020; Lu et al., 2017). For the i-th random variable, the inverse transformation is written as

$$X_i = \theta_{i1} + \theta_{i2} H_i(Z_i), \qquad H_i(Z_i) = a_{2i} Z_i^2 + a_{1i} Z_i - a_{2i}, \tag{13}$$

where θ_{i1} and θ_{i2} are the mean and standard deviation, respectively; Z_i is a correlated standard normal variable; and a_{1i} and a_{2i} are determined by the fixed skewness α_{3i}. The correlated standard normal vector is obtained from the independent standard normal vector by

$$\boldsymbol{Z} = \boldsymbol{L}_0 \boldsymbol{U}, \tag{14}$$

where $\boldsymbol{L}_0$ is obtained from the Cholesky decomposition of $\boldsymbol{C}_Z$. For X_i and X_j, the physical correlation coefficient λ_{ij} and the transformed correlation coefficient λ_{0ij} satisfy

$$\lambda_{ij} = \Delta_{ij}\lambda_{0ij} + \frac{3}{50}\alpha_{3i}\alpha_{3j}\lambda_{0ij}^2, \quad \Delta_{ij} = \sqrt{\left(1 - \frac{3}{50}\alpha_{3i}^2\right)\left(1 - \frac{3}{50}\alpha_{3j}^2\right)}. \tag{15}$$

If $\alpha_{3i}\alpha_{3j} = 0$, $\lambda_{0ij} = \lambda_{ij}/\Delta_{ij}$; otherwise,

$$\lambda_{0ij} = \frac{-25\Delta_{ij} + 5\sqrt{25\Delta_{ij}^2 + 6\alpha_{3i}\alpha_{3j}\lambda_{ij}}}{3\alpha_{3i}\alpha_{3j}}. \tag{16}$$

To determine $(\boldsymbol{\Theta}_{\text{low}}, \boldsymbol{\Lambda}_{\text{low}})$ and $(\boldsymbol{\Theta}_{\text{up}}, \boldsymbol{\Lambda}_{\text{up}})$, the changing tendency of $P_f(\boldsymbol{\Theta}, \boldsymbol{\Lambda})$ with respect to each interval parameter needs to be investigated. Let η denote one component of $\boldsymbol{\Theta}$ or $\boldsymbol{\Lambda}$. Based on Eq. (8), since $\phi_{\boldsymbol{U}}(\boldsymbol{u})$ is not related to η, the derivative of failure probability with respect to η can be expressed as

$$\frac{\partial P_f}{\partial \eta} = \int_{\Omega_{\boldsymbol{U}}} \frac{\partial I_{\boldsymbol{U}}(\boldsymbol{u} \mid \boldsymbol{\Theta}, \boldsymbol{\Lambda})}{\partial \eta} \phi_{\boldsymbol{U}}(\boldsymbol{u}) \mathrm{d}\boldsymbol{u}. \tag{17}$$

Then, based on Eq. (9) and using the chain rule of differentiation, $\partial I_{\boldsymbol{U}}/\partial \eta$ is written as

$$\frac{\partial I_{\boldsymbol{U}}(\boldsymbol{U} \mid \boldsymbol{\Theta}, \boldsymbol{\Lambda})}{\partial \eta} = \frac{\partial I_{\boldsymbol{X}}[S(\boldsymbol{U} \mid \boldsymbol{\Theta}, \boldsymbol{\Lambda})]}{\partial S(\boldsymbol{U} \mid \boldsymbol{\Theta}, \boldsymbol{\Lambda})} \frac{\partial S(\boldsymbol{U} \mid \boldsymbol{\Theta}, \boldsymbol{\Lambda})}{\partial \eta}. \tag{18}$$

The derivative of the indicator function is nonzero only on the limit state surface $G[S(\boldsymbol{U} \mid \boldsymbol{\Theta}, \boldsymbol{\Lambda})] = 0$. Substituting Eq. (18) into Eq. (17) gives

$$\frac{\partial P_f}{\partial \eta} = \int_{G[S(\boldsymbol{U}|\boldsymbol{\Theta},\boldsymbol{\Lambda})]=0} \frac{\partial I_{\boldsymbol{X}}[S(\boldsymbol{U} \mid \boldsymbol{\Theta}, \boldsymbol{\Lambda})]}{\partial S(\boldsymbol{U} \mid \boldsymbol{\Theta}, \boldsymbol{\Lambda})} \frac{\partial S(\boldsymbol{U} \mid \boldsymbol{\Theta}, \boldsymbol{\Lambda})}{\partial \eta} \phi_{\boldsymbol{U}}(\boldsymbol{u}) \mathrm{d}\boldsymbol{u}. \tag{19}$$

Since $\phi_{\boldsymbol{U}}(\boldsymbol{u})$ is always positive, the sign of $\partial P_f/\partial \eta$ is determined by the sign of the two derivative terms in Eq. (19). Based on the definition of the indicator function, the sign of the first term is

$$\operatorname{sign}\left[\frac{\partial I_{\boldsymbol{X}}[S(\boldsymbol{U} \mid \boldsymbol{\Theta}, \boldsymbol{\Lambda})]}{\partial S(\boldsymbol{U} \mid \boldsymbol{\Theta}, \boldsymbol{\Lambda})}\right] = -\operatorname{sign}\left[\frac{\partial G(\boldsymbol{X})}{\partial \boldsymbol{X}^{\mathsf{T}}}\right]. \tag{20}$$

Therefore, for a monotonic performance function, the correspondence between failure probability bounds and uncertain parameters can be obtained by analyzing $\partial S/\partial\eta$ for the uncertain first moment, second moment, and correlation coefficient.

3.2.1. *Changing Tendency of Failure Probability with Mean*

For the first moment of X_i, the interval parameter is θ_{i1}. According to Eq. (13), the i-th component of the inverse transformation is

$$S_i = \theta_{i1} + \theta_{i2} H_i(Z_i). \tag{21}$$

The derivative of S_i with respect to θ_{i1} is

$$\frac{\partial S_i}{\partial \theta_{i1}} = 1. \tag{22}$$

Substitution of Eqs. (20) and (22) into Eq. (19) leads to

$$\text{sign}\left[\frac{\partial P_f}{\partial \theta_{i1}}\right] = -\text{sign}\left[\frac{\partial G(\boldsymbol{X})}{\partial X_i}\right]. \tag{23}$$

Thus, if $\partial G(\boldsymbol{X})/\partial X_i < 0$, P_f increases with θ_{i1}; if $\partial G(\boldsymbol{X})/\partial X_i \geq 0$, P_f decreases with θ_{i1}. The corresponding relationship between the failure probability bounds and θ_{i1} is summarized in Table 1.

3.2.2. *Changing Tendency of Failure Probability with Standard Deviation*

For the second moment of X_i, the interval parameter is θ_{i2}. From Eq. (21), the derivative of S_i with respect to θ_{i2} is

$$\frac{\partial S_i}{\partial \theta_{i2}} = H_i(Z_i) = a_{2i} Z_i^2 + a_{1i} Z_i - a_{2i}. \tag{24}$$

For the marginal correspondence rule, the standard coordinate entering $H_i(\cdot)$ is denoted by U_i. Combining Eqs. (20) and (24), the sign of the derivative of failure probability is

$$\text{sign}\left[\frac{\partial P_f}{\partial \theta_{i2}}\right] = -\text{sign}\left[\frac{\partial G(\boldsymbol{X})}{\partial X_i} H_i(U_i)\right]. \tag{25}$$

When the skewness is zero, $a_{1i} = 1$ and $a_{2i} = 0$, so $H_i(U_i) = U_i$. When the skewness is nonzero, the sign of $H_i(U_i)$ changes at the roots p and q, which are expressed as

$$p = \frac{-5}{\sqrt{3}\alpha_{3i}} - \frac{5}{\sqrt{3}\left|\alpha_{3i}\right|}, \quad q = \frac{-5}{\sqrt{3}\alpha_{3i}} + \frac{5}{\sqrt{3}\left|\alpha_{3i}\right|}. \tag{26}$$

For $\alpha_{3i} < 0$,

$$\begin{cases} H_i(U_i) < 0, \ U_i \in (-\infty, 0) \cup (q, +\infty), \\ H_i(U_i) \geq 0, \ U_i \in [0, q], \end{cases} \tag{27}$$

and for $\alpha_{3i} > 0$,

$$\begin{cases} H_i(U_i) < 0, \ U_i \in [p, 0], \\ H_i(U_i) \geq 0, \ U_i \in (-\infty, p) \cup (0, +\infty). \end{cases} \tag{28}$$

The sign of $\partial P_f/\partial\theta_{i2}$ is then determined jointly by $\partial G(\boldsymbol{X})/\partial X_i$ and the interval of U_i. The resulting relationship between the failure probability bounds and θ_{i2} is summarized in Table 1.

3.2.3. *Changing Tendency of Failure Probability with Correlation Coefficient*

For an uncertain correlation coefficient, the interval parameter is λ_{ij}. Unlike the first two moments, λ_{ij} changes the transformation from the independent standard normal vector $\boldsymbol{U}$ to the correlated standard normal vector $\boldsymbol{Z}$. For the pair (X_i, X_j), the Cholesky transformation gives

$$Z_i = U_i, \qquad Z_j = \lambda_{0ij}U_i + \sqrt{1-\lambda_{0ij}^2}\,U_j. \tag{29}$$

Therefore,

$$\frac{\partial Z_j}{\partial\lambda_{0ij}} = U_i - \frac{\lambda_{0ij}}{\sqrt{1-\lambda_{0ij}^2}}U_j = U_i - k_{ij}U_j, \qquad k_{ij} = \frac{\lambda_{0ij}}{\sqrt{1-\lambda_{0ij}^2}}. \tag{30}$$

The derivative of the inverse transformation with respect to λ_{ij} can be decomposed as

$$\frac{\partial S}{\partial\lambda_{ij}} = \frac{\partial S}{\partial\boldsymbol{Z}^{\mathsf{T}}}\frac{\partial\boldsymbol{Z}^{\mathsf{T}}}{\partial\lambda_{0ij}}\frac{\partial\lambda_{0ij}}{\partial\lambda_{ij}}. \tag{31}$$

For the simplified third-moment normal transformation, the relevant component of $\partial S/\partial\boldsymbol{Z}^{\mathsf{T}}$ is positive:

$$\frac{\partial S_j}{\partial Z_j} = \theta_{j2}\left(2a_{2j}Z_j + a_{1j}\right) > 0. \tag{32}$$

The derivative from λ_{ij} to λ_{0ij} is also positive. If $\alpha_{3i}\alpha_{3j} = 0$,

$$\frac{\partial\lambda_{0ij}}{\partial\lambda_{ij}} = \frac{1}{\Delta_{ij}} > 0. \tag{33}$$

Otherwise,

$$\frac{\partial\lambda_{0ij}}{\partial\lambda_{ij}} = \frac{5}{\sqrt{25\Delta_{ij}^2 + 6\alpha_{3i}\alpha_{3j}\lambda_{ij}}} > 0. \tag{34}$$

Substituting Eqs. (30)–(34) into Eq. (19), the sign of the derivative of failure probability with respect to λ_{ij} is obtained as

$$\operatorname{sign}\left[\frac{\partial P_f}{\partial\lambda_{ij}}\right] = -\operatorname{sign}\left[\frac{\partial G(\boldsymbol{X})}{\partial X_j}\left(U_i - k_{ij}U_j\right)\right]. \tag{35}$$

When λ_{ij} is an interval variable, λ_{0ij} and k_{ij} are also interval quantities. The two threshold values used for determining the sign of $U_i - k_{ij}U_j$ are

$$\underline{k}_{0ij} = \frac{\underline{\lambda}_{0ij}U_j}{\sqrt{1-\underline{\lambda}_{0ij}^2}}, \quad \overline{k}_{0ij} = \frac{\overline{\lambda}_{0ij}U_j}{\sqrt{1-\overline{\lambda}_{0ij}^2}}. \tag{36}$$

Table I. Values of uncertain moments and correlation coefficients for inverse normal transformation

Uncertain parameter	$\dfrac{\partial G(\boldsymbol{X})}{\partial X_j}$	$\dfrac{\partial G(\boldsymbol{X})}{\partial X_i}$	U_i		$\boldsymbol{\Theta}_{\text{low}}/\boldsymbol{\Lambda}_{\text{low}}$	$\boldsymbol{\Theta}_{\text{up}}/\boldsymbol{\Lambda}_{\text{up}}$
			$\alpha_{3i} \leq 0$	$\alpha_{3i} > 0$		
θ_{i1}		< 0	$(-\infty, +\infty)$		$\underline{\theta}_{i1}$	$\overline{\theta}_{i1}$
		≥ 0			$\overline{\theta}_{i1}$	$\underline{\theta}_{i1}$
θ_{i2}		< 0	$(-\infty, 0) \cup (q, +\infty)$	$[p, 0]$	$\overline{\theta}_{i2}$	$\underline{\theta}_{i2}$
			$[0, q]$	$(-\infty, p) \cup (0, +\infty)$	$\underline{\theta}_{i2}$	$\overline{\theta}_{i2}$
		≥ 0	$(-\infty, 0) \cup (q, +\infty)$	$[p, 0]$	$\underline{\theta}_{i2}$	$\overline{\theta}_{i2}$
			$[0, q]$	$(-\infty, p) \cup (0, +\infty)$	$\overline{\theta}_{i2}$	$\underline{\theta}_{i2}$
λ_{ij}	≥ 0		$(-\infty, \overline{k}_{0ij}]$		$\underline{\lambda}_{ij}$	$\overline{\lambda}_{ij}$
			$(\overline{k}_{0ij}, +\infty)$		$\overline{\lambda}_{ij}$	$\underline{\lambda}_{ij}$
	< 0		$(-\infty, \underline{k}_{0ij}]$		$\overline{\lambda}_{ij}$	$\underline{\lambda}_{ij}$
			$(\underline{k}_{0ij}, +\infty)$		$\underline{\lambda}_{ij}$	$\overline{\lambda}_{ij}$

After evaluating $\partial G(\boldsymbol{X})/\partial X_j$ and the position of U_i relative to the threshold interval, the correspondence between the failure probability bounds and λ_{ij} can be obtained. The resulting relationships for the first moment, second moment, and correlation coefficient are summarized in Table 1.

The corresponding parameter values define two transformed limit state functions. One corresponds to the lower failure probability bound, and the other corresponds to the upper bound. Since the standard normal density is independent of the interval parameters, the remaining calculation can be carried out by a conventional reliability method.

3.3. Computational Procedure

With $(\boldsymbol{\Theta}_{\text{low}}, \boldsymbol{\Lambda}_{\text{low}})$ and $(\boldsymbol{\Theta}_{\text{up}}, \boldsymbol{\Lambda}_{\text{up}})$ determined using Table 1, the distributions with uncertain moments and correlation coefficients can be replaced by general distributions for the bounds of failure probability. Then, β^I and P_f^I can be computed using classical reliability analysis methods, such as the first-order reliability method (FORM) (Hasofer and Lind, 1974; Rackwitz and Fiessler, 1978), the second-order reliability method (SORM) (Zhao and Ono, 1999; Breitung, 2021), and the method of moments (Zhao and Lu, 2021).

The proposed method for evaluating β^I and P_f^I consists of two main steps:

(1) Calculate the partial derivatives $\partial G(\boldsymbol{X})/\partial x_j$ for random variables related to uncertain statistical moments or uncertain correlation coefficients. For implicit performance functions, these derivatives can be evaluated using finite differences.

(2) Calculate β^I and P_f^I using appropriate reliability methods. The bounds are obtained through two separate reliability analyses.

The computational cost of the proposed method is measured by the number of performance function evaluations. For explicit performance functions, n evaluations are required to calculate the partial derivatives; for implicit functions, $n + 1$ evaluations are needed, where n represents the number of random variables related to uncertain statistical moments or uncertain correlation coefficients. The determination of the moment and correlation coefficient bounds requires no additional performance function evaluations. The method then requires two classical reliability analyses, making the total computational cost comparable to that of two reliability analyses. This computational cost is practical for engineering applications in which each performance function evaluation may be expensive.

4. Illustrative Examples

Two examples are used to illustrate the proposed method. All examples include uncertain first two moments and uncertain correlation coefficients. The first example considers a mathematical problem with analytical solution, which is adopted to examine the accuracy of the proposed method. The second example applies the method to an annular cross-section column.

4.1. Linear Performance Function

The first example uses the linear performance function

$$G(\mathbf{X}) = X_1 - X_2. \tag{37}$$

The two variables are normal variables. Because only limited information is assumed to be available, the means, standard deviations, and the correlation coefficient are represented by intervals. The input information is listed in the following table.

Table II. Probabilistic information for Example 1.

Variable	Mean	Standard deviation	Distribution	Correlation coefficient
X_1	$[45, 55]$	$[6.3, 7.7]$	Normal	$[0.6, 0.8]$
X_2	$[31.5, 38.5]$	$[4.5, 5.5]$	Normal	

For this performance function, $\partial G/\partial X_1 = 1$ and $\partial G/\partial X_2 = -1$. The bounds of reliability index are shown in Table III. It is found that, the results of the proposed method combined with FORM can obtain exact evaluation of β^I, while the results from the proposed method combined with method of moments has a relatively small R.E. within 5%.

4.2. Annular Cross-Section Column

The second example considers an annular cross-section column under axial load. The performance function is adapted from Ref. (Zhang and Xu, 2022) and is written as

$$G(\mathbf{X}) = \frac{\pi^2 E}{L^2}\left[\frac{\pi}{64}\left((D+T)^4 - D^4\right)\right] - F, \tag{38}$$

Table III. Reliability analysis results for Example 1.

Method	Lower bound of β		Upper bound of β		Function evaluations
	Value	R.E. (%)	Value	R.E. (%)	
Analytical solution	0.5834	–	3.4816	–	–
Proposed method-FORM	0.5834	–	3.4816	–	6
Proposed method-MoM	0.5728	1.81	3.5934	3.21	52

where E is the elastic modulus, T is the thickness, D is the internal diameter, L is the column length, and F is the axial load. The variables T and D are treated as random variables with interval means and interval standard deviations. Their correlation coefficient is also represented by an interval, $\lambda_{TD} = [-0.4, -0.1]$.

Table IV. Probabilistic information for Example 2.

Variable	Description	Distribution	Mean	Standard deviation
E (MPa)	Elastic modulus	Lognormal	203000	30450
T (mm)	Thickness	Normal	$[3.5, 4.5]$	$[0.095, 0.105]$
D (mm)	Internal diameter	Normal	$[23, 24]$	$[0.15, 0.25]$
L (mm)	Length of column	Normal	2500	50
F (N)	Axial load	Lognormal	3000	550

In this example, $\partial G/\partial T$, and $\partial G/\partial D$ are positive. The calculated bounds of reliability index are listed in Table V. It can be observed from Table V that, the relative error between the results of the proposed method combined with the method of moments and those from MCS remains under 4%, demonstrating the accuracy of the proposed method.

Table V. Reliability analysis results for Example 2.

Method	Lower bound of β		Upper bound of β		Function evaluations
	Value	R.E. (%)	Value	R.E. (%)	
Double-loop MCS	0.4590	–	2.2888	–	–
Proposed method	0.4459	2.86	2.3630	3.24	782

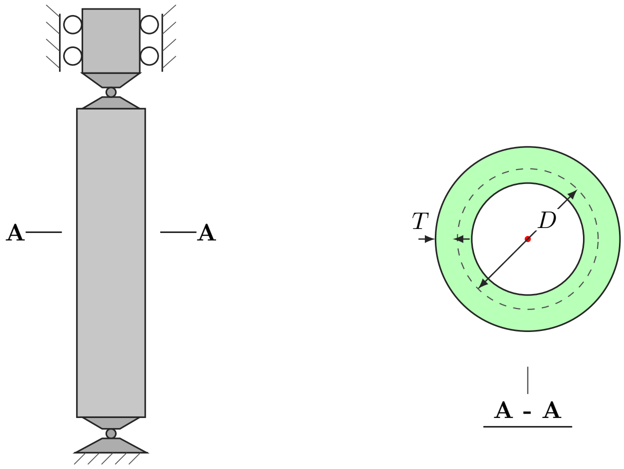

Figure 1. Annular section column.

5. Conclusions

This paper presents a hybrid reliability analysis method for problems in which the first two statistical moments and the correlation coefficients of random variables are both uncertain. The uncertain means, standard deviations, and correlation coefficients are described by interval variables. Based on the derivative of the performance function with respect to the random variables with uncertain parameters, the bounds of the failure probability are then obtained by two traditional reliability analyses.

The main advantage of the proposed method is that it avoids the need for optimization or sampling. The examples show that the proposed method can efficiently calculate the bounds of failure probability, which are in close agreement with those obtained through double-loop MCS.

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