Stochastic contact analysis and its applications

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Abstract. Stochastic contact problems (SCPs) arise in many engineering applications where uncertainties in material properties, geometry or loading conditions affect contact behavior. Compared with deterministic contact analysis, the coupling of contact mechanics and randomness dramatically increases the computational complexity. In particular, stochastic contact nodes whose probability density functions can be strongly discontinuous or even singular render SCPs highly sensitive to uncertain inputs and computationally demanding. To address these challenges, a unified framework is presented for solving SCPs. It integrates Lagrange multiplier, penalty or augmented Lagrange multiplier formulations with semi-reduced-order approximations. In this framework, stochastic solutions are approximated as sums of products of random variables and deterministic vectors, thereby transforming high-dimensional SCPs into deterministic problems coupled with low-dimensional stochastic algebraic systems. Semi-reduced-order equations further combine smooth reduced-order solutions with full-order nonsmooth components to reduce computational cost without sacrificing accuracy. In addition, the framework is extended to rolling SCPs via the development of a stochastic arbitrary Lagrangian–Eulerian (ALE) formulation, which enables efficient analysis of rolling interfaces with uncertainties. Numerical results on high-dimensional SCP benchmarks and engineering-level cases such as rolling tires demonstrate that the proposed framework is both efficient and scalable. It accurately captures stochastic contact nodes while also significantly reducing computational cost. This framework provides a powerful tool for robust design and reliability assessment of engineering contact systems under uncertainty.

Keywords: Stochastic contact; Stochastic Lagrangian multiplier/penalty methods; Rolling contact analysis; Stochastic arbitrary Lagrangian–Eulerian method.

1. Introduction

Contact problems play a critical role in engineering practice (Wriggers and Laursen, 2006). As uncertainties are unavoidable in many problems, the consideration of uncertainties in system behavior has become an essential part of the analysis and design of engineering applications. However, there is limited experience in the treatment of SCPs. This article focuses on two-body contact problems with random parameters, e.g., random material properties and external forces. Currently, there are several methods for solving SCPs. The Monte Carlo (MC) method and its extensions such as the multilevel Monte Carlo (MLMC) method and the adaptive MLMC method (Chevalier et al., 2005; Bierig and Chernov, 2015; Rey et al., 2019) have been applied to SCPs. Although MC-type methods can be readily implemented using existing deterministic codes, a large number of deterministic contact problems must be solved to achieve high-accuracy solutions, which is prohibitively expensive for large-scale problems in practice.

Another approach is the polynomial chaos (PC)-based method (Ghanem and Spanos, 2003; Xiu and Karniadakis, 2002). In this method, the stochastic solution is projected onto a stochastic space spanned by (generalized) PC basis, and the stochastic Galerkin projection is used to transform the original stochastic problem into a deterministic equation that is much larger than the original stochastic problem. However, its accuracy and efficiency still need improvement. On one hand, the classical PC method cannot accurately capture discontinuous stochastic contact solutions. Hence it usually works in conjunction with other methods such as the stochastic collocation method (Forster and Kornhuber, 2010; Arnst and Ghanem, 2012). On the other hand, because the computational effort increases sharply with the number of random variables and the expansion order, the PC method suffers from the so-called curse of dimensionality (Arnst and Ghanem, 2012), which is prohibitively expensive for large-scale and high-dimensional cases. Furthermore, other methods have also been proposed to solve contact-related problems, for example, within the framework of stochastic variational inequality problems (Jiang and Xu, 2008; Garreis and Ulbrich, 2017).

To solve SCPs efficiently and accurately, a unified framework is proposed to integrate Lagrange multiplier, penalty and augmented Lagrange multiplier formulations with semi-reduced-order approximations. The stochastic solutions are approximated as sums of products of random variables and deterministic vectors, thereby transforming high-dimensional SCPs into deterministic problems coupled with low-dimensional stochastic algebraic systems. They are then computed in their individual spaces. The deterministic problems are solved by existing finite element methods. The stochastic algebraic equations are solved by a sampling method, and the computational cost does not increase dramatically with the stochastic dimension. Hence, the curse of dimensionality in high-dimensional stochastic spaces is effectively circumvented. The effectiveness of the proposed approaches is demonstrated on numerical examples of up to 100 dimensions. The full-reduced-order equations directly constructed from the reduced basis cannot adequately capture the nonsmooth solutions of stochastic contact problems. Therefore, semi-reduced-order equations are used to combine smooth reduced-order solutions with full-order nonsmooth components, reducing computational cost without sacrificing accuracy. The size of the semi-reduced-order equation is small since the possible contact region is small compared to the whole domain in many applications. In addition, the framework is extended to rolling SCPs by developing a stochastic ALE formulation, which enables efficient analysis of rolling interfaces with uncertainties.

2. Stochastic Lagrange multiplier method and stochastic penalty method

For the treatment of randomness in structural parameters, e.g., material properties or potential contact points within the framework of the finite element method (FEM), we propose reduced-order computational schemes for both the Lagrange multiplier (LM) method and the penalty method (PM) to enforce the contact constraints (Zheng and Nackenhorst, 2023). In order to reduce the computational effort, in particular for high-dimensional stochastic problems, e.g., random fields for material properties, we suggest splitting the solution into pure deterministic computation and an iterative scheme to solve the stochastic problem. Both schemes will be compared to the native Monte Carlo (MC) method with regard to accuracy and computational effort.

For the LM approach we split the solution into pure deterministic vectors and random variables

$$\mathbf{u}_k(\theta) = \mathbf{u}_{k-1} + \lambda_k(\theta)\,\mathbf{d}_k, \tag{1}$$

where $\mathbf{d}_k$ are solutions of the designed deterministic contact problems and $\lambda_k(\theta)$ are associated random variables. As given in (Zheng and Nackenhorst, 2023), the solution is obtained by an iterative scheme solving for $\mathbf{d}_k$ and $\lambda_k(\theta)$ in a staggered way. By doing so, the possibly high-dimensional problem is decomposed into solutions of deterministic contact problems and one-dimensional stochastic solutions, and the latter can be efficiently solved using sampling processes, even for very high-dimensional random inputs.

Because of the non-smoothness of the solutions, particularly for contact problems, we have observed that the accuracy of the solution can be improved by a semi-reduced model order reduction approach. Reduced-order modeling is applied only to non-contact nodes. Thus, the degrees of freedom of the active contact nodes are retained for solving the original finite element problem.

Unfortunately, the previously outlined approach does not work for the PM approach, because the typically large penalty parameter numerically deteriorates the update loop for the random variables. Here, an alternative scheme is proposed using the incremental solution

$$\mathbf{u}_k(\theta) = \mathbf{u}_{k-1}(\theta) + \Delta\mathbf{u}_k(\theta)\,, \tag{2}$$

where the stochastic increment $\Delta\mathbf{u}_k(\theta) = [\lambda_{k,1}(\theta)\,\mathbf{Z}_1 + \lambda_{k,2}(\theta)\,\mathbf{Z}_2]\,\mathbf{d}_k$, $\lambda_{k,1}(\theta)$ and $\lambda_{k,2}(\theta)$ are scalar random variables, and $\mathbf{d}_k$ is the solution vector of the designed deterministic contact problems. Similar to the semi-reduced approach for the LM-based technique, the matrices $\mathbf{Z}_1$ and $\mathbf{Z}_2$ are index matrices, which split the finite element matrix into non-contact and contact nodes. A staggered iterative scheme is developed to solve $\lambda_{k,1}(\theta)$, $\lambda_{k,2}(\theta)$ and $\mathbf{d}_k$.

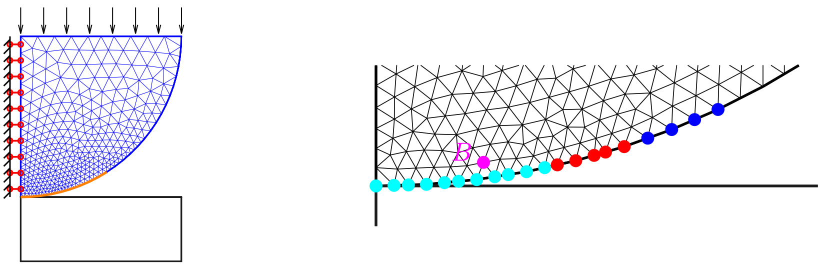

Figure 1. Model and its finite element mesh (left) and nodes on the possible contact interface (right).

The proposed methods are tested using a stochastic contact problem between a hemisphere and a rigid body as shown in Figure 1 (left). The Young's modulus is modeled as a random field that is approximated by Karhunen-Loève expansion with 10 and 100 random variables. As shown in Figure 1 (right), there are three types of characteristic nodes on the possible contact interface, including the contact nodes (cyan points), the possible contact nodes (red points), and the non-contact nodes (blue points), where the possible contact nodes are the nodes that are in contact states for some samples of random parameters and in non-contact states for other samples, and they are thus referred to as "possible" (or stochastic) contact nodes.

Probability density functions (PDFs) of a possible contact node and the reference point B not on the contact interface (shown in Figure 1 (right)) calculated via the LM, PM and native MC approaches are shown in Figure 2 (left), (center) and (right), respectively. Both the LM and PM methods can capture strongly discontinuous and singular PDFs and achieve comparable accuracy to the native MC method. Computational times for stochastic dimensions of 10 and 100 are listed in Table I. The proposed methods can solve both low- and high-dimensional stochastic contact problems at low cost and are much more efficient than the MC method.

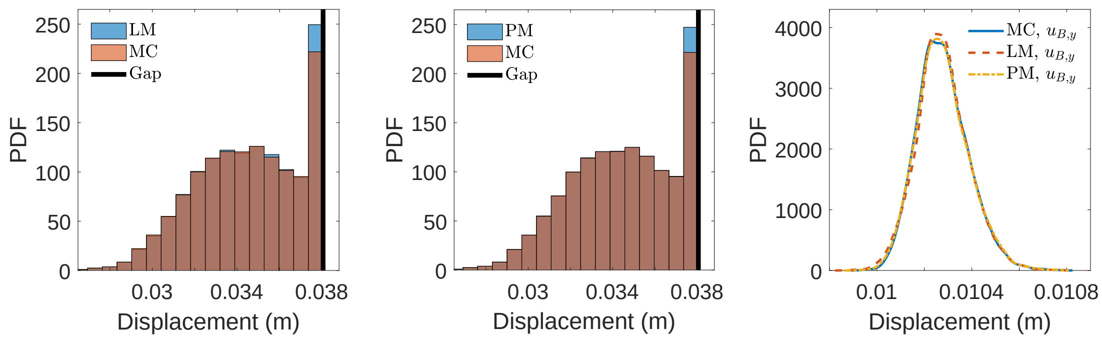

Figure 2. PDFs of a possible contact node obtained by MC, LM (left) and PM (mid) methods and PDFs of the vertical displacement of the reference point B (right).

Table I. Computational costs (seconds) for stochastic dimensions of 10 and 100.

Dimension	LM	PM	MCS
10	51	37	1742
100	241	269	2004

3. Stochastic augmented Lagrange multiplier method

As is well known in deterministic contact analysis, the augmented Lagrange multiplier method offers better robustness and convenience. Following this, we extend the stochastic Lagrange multiplier method and the stochastic penalty method presented above to stochastic augmented Lagrange multiplier methods. To this end, we still employ the stochastic approximation (2). New stochastic Uzawa-type algorithms are proposed to solve SCPs, and they also provide new insights into SCPs through the so-called stochastic contact interface systems (Zheng and Nackenhorst, 2025b).

Analogous to the deterministic Uzawa approach (Wriggers and Laursen, 2006; Laborde and Renard, 2008), stochastic contact constraints are imposed by weak penalties and stochastic Lagrange multipliers. Based on the stochastic solution approximation described above, a stochastic Uzawa algorithm and a generalized stochastic Uzawa algorithm are presented to solve for the stochastic Lagrange multipliers and each pair of random variable and deterministic vector in a greedy manner. In the stochastic Uzawa algorithm, the stochastic Lagrange multipliers are updated through a randomized Uzawa approach based on the known pair of random variable and deterministic vector (or given by initial values). Based on the obtained stochastic Lagrange multipliers, each pair of random variable and deterministic vector is then solved using a local alternating iteration. In the local alternating iteration, the original SCP is transformed into deterministic problems and two-dimensional stochastic algebraic equations. The deterministic problems can be solved by existing numerical solvers, and the two-dimensional stochastic equations are solved by a non-intrusive sampling method. Since the sampling method is not sensitive to stochastic dimensions, the proposed method can be applied to high-dimensional SCPs without further modifications. The generalized stochastic Uzawa algorithm avoids the local alternating iteration and only requires a three-component alternating iteration. One component of the random variable, the deterministic vector, or the stochastic Lagrange multipliers is solved by assuming the other two are known, where the stochastic Lagrange multipliers are updated through a randomized Uzawa approach, and the random variable and the deterministic vector are solved using two-dimensional stochastic algebraic equations and deterministic problems that are obtained in a manner similar to the local alternating iteration described above.

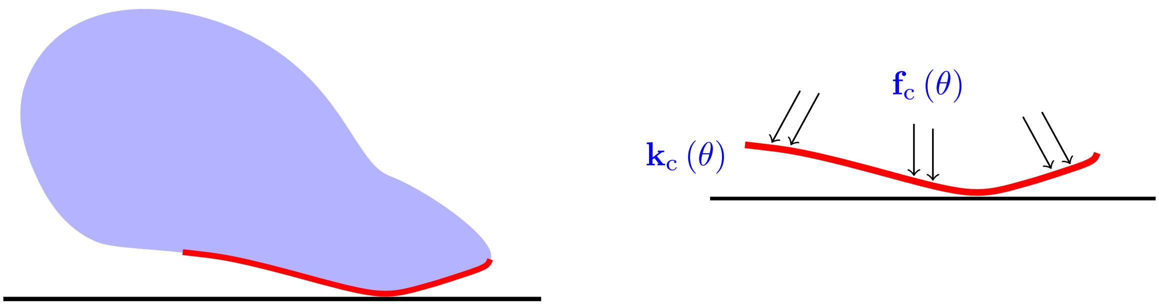

Figure 3. The flexible-rigid stochastic contact problem (left) and the corresponding SCI system (right).

To further improve the accuracy, the stochastic solution is recalculated by an equivalent stochastic contact interface system, which is generated by representing the non-contact stochastic solution of the original SCP using the contact stochastic solution and the obtained deterministic vectors, as illustrated in Figure 3. The stochastic contact interface system involves only the contact stochastic solution. Its size is much smaller than that of the original SCP, since the possible contact region is usually small compared to the whole domain in most cases. In this way, the SCP is solved only on the contact interface. Therefore, good convergence and high accuracy can be achieved.

As a test, we consider a 1D Euler–Bernoulli beam fixed on the left and constrained by a rigid block on the right, as shown in Figure 4. Both the SU and GSU methods are used to solve this example. The iterative errors of the SU and GSU methods are shown in Figure 5 (left). Three retained terms are sufficient for both methods, which demonstrates the good convergence of the proposed methods. In this case, the SU and GSU methods have very similar convergence properties. To check the computational accuracy of the proposed method, the PDFs of the vertical displacements of the contact node obtained by the SU method and MC simulations are compared in Figure 5 (right). Note that the PDF is discontinuous at the boundaries 0.01 and 0.015 of the random gap. As can be seen from Figure 5, the proposed method captures the discontinuities very well. The computational times of the two methods are listed in Table II. It can be seen that both methods are much more efficient than MC simulations. Since the GSU method avoids the local alternating iteration used in the SU method, its iteration time is lower than that of the SU method. Furthermore, the same number of deterministic vectors results in similar recalculation times for the two methods.

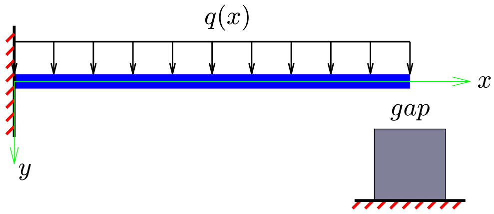

Figure 4. Euler–Bernoulli beam supported by a rigid block with a random gap.

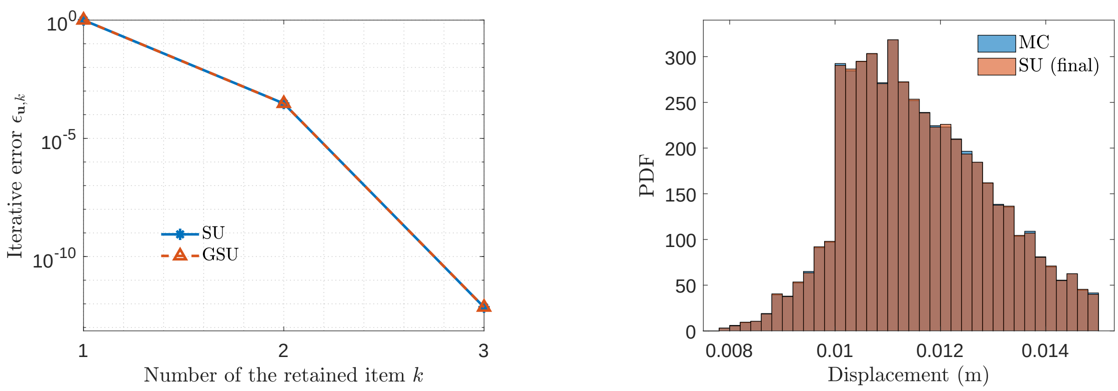

Figure 5. Iterative errors of different retained terms (left) and the PDFs of the contact point (right).

Table II. Computational times (seconds) of the SU and GSU methods.

Method	Iteration time	Recalculation time	Total time	MCS
SU	0.52	0.89	1.41	23.51
GSU	0.09	0.94	1.03	

4. Stochastic rolling contact analysis

In this part, we consider rolling contact on a rigid surface, which occurs widely in situations such as tire rolling contact. Following the idea of the deterministic ALE method for rolling contact analysis (Nackenhorst, 2004), the stochastic rolling motion is decomposed into pure rigid body motion and its stochastic deformation. As shown in Figure 6, the initial configuration is Ω_0, and the reference and current configurations are given by Ω_χ and Ω_φ, respectively. With these, we present a stochastic ALE method to handle rolling contact with uncertain inputs (Zheng and Nackenhorst, 2025a), e.g., material properties, velocity and external forces. The following stochastic finite element equation is thus generated

$$\left[\mathbf{K}_0(\theta) - \mathbf{W}(\theta) + \alpha \mathbf{K}_\mathrm{p}(\theta)\right] \mathbf{u}(\theta) = \mathbf{F}(\theta) + \alpha \mathbf{F}_\mathrm{p}(\theta), \tag{3}$$

where the stochastic global stiffness matrix $\mathbf{K}_0(\theta)$, the stochastic global inertia matrix $\mathbf{W}(\theta)$ (including the contribution of the stochastic flux term), the stochastic global contact stiffness matrix $\mathbf{K}_\mathrm{p}(\theta)$, the stochastic global force vector $\mathbf{F}(\theta)$, and the stochastic global contact force vector $\mathbf{F}_\mathrm{p}(\theta)$ are assembled using their element matrices and vectors (Nackenhorst, 2004; Zheng and Nackenhorst, 2025a). For efficient solutions, we use the stochastic penalty method and still employ the stochastic approximation (2).

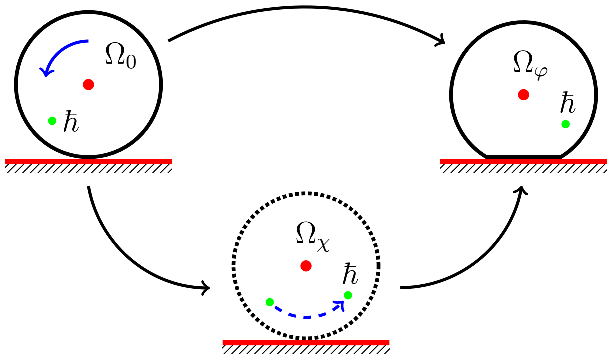

Figure 6. SALE framework for stochastic rolling contact.

A 2D rolling model with material and velocity uncertainties is used to illustrate the proposed method, as shown in Figure 7. The PDFs of stochastic contact nodes are shown in Figure 8 and are compared with MC simulations, demonstrating good accuracy in capturing discontinuous PDFs.

5. Conclusion and discussion

This paper presents a framework to handle SCPs by integrating Lagrange multiplier, penalty and augmented Lagrange multiplier formulations with semi-reduced-order approaches. In this framework, SCPs are decoupled into deterministic problems and low-dimensional stochastic algebraic systems. Semi-reduced-order equations further reduce computational cost by combining smooth reduced-order solutions with full-order nonsmooth components. Numerical results demonstrate the

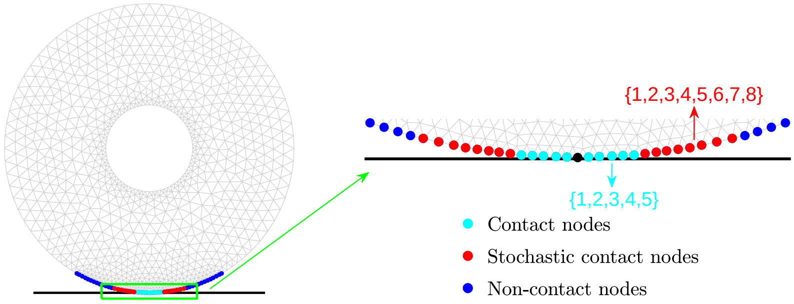

Figure 7. Contact nodes, stochastic contact nodes and non-contact nodes.

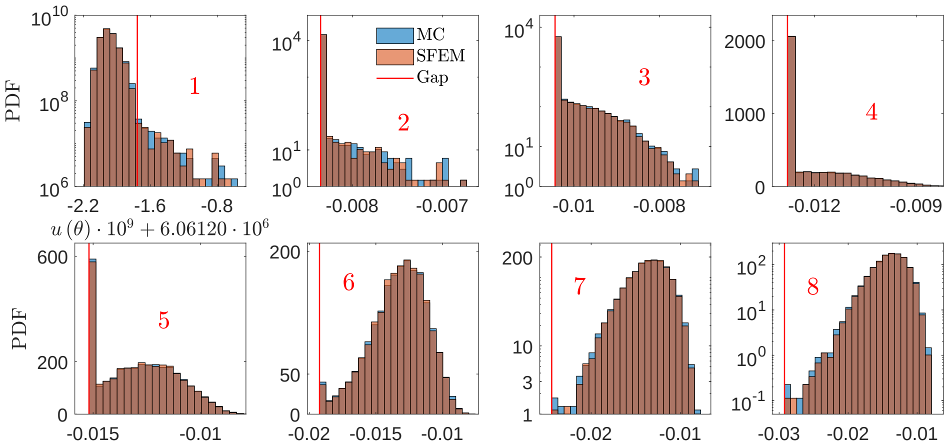

Figure 8. PDFs of the stochastic displacements in the y direction of the stochastic contact nodes.

promising performance of the proposed framework. It can accurately capture discontinuous PDFs of stochastic contact nodes. Although the framework is general-purpose, it still requires further investigation to extend the method to more complex stochastic contact analysis, such as stochastic contact with large deformation and plasticity.

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